19 Dynamic Optimization Under Uncertainty: Non-convex Feasible Set

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[The] object [of the book] is refinement, not reconstruction; it is a study in 'pure theory'. The motive back of its presentation is twofold. In the first place, the writer cherishes, in the face of the pragmatic, philistine tendencies of the present age, especially characteristic of the thought of our own country, the hope that careful, rigorous thinking in the field of social problems does after all have some significance for human weal and woe. In the second place, he has a feeling that the 'practicalism' of the times is a passing phase, even to some extent, a pose; that there is a strong undercurrent of discontent with loose and superficial thinking and a real desire, out of sheer intellectual self-respect, to reach a clearer understanding of the meaning of terms and dogmas which pass current as representing ideas. (Frank H. Knight, in *Risk Uncertainty and Profit*)

1 INTRODUCTION

An editorial note in the *Economic Journal* (May 1930) reported the death of Frank Ramsey, and his 1928 paper was described as 'one of the most remarkable contributions to mathematical economics ever made'. In the same issue the editor organized a symposium on increasing returns and the representative firm. This symposium seems to be a natural follow-up of a number of papers published by the Journal during 1926–8, including the well-known article of Allyn Young (1928) that is still available, and duly remembered. The problems of equilibrium of a firm under increasing returns, or more generally, of designing price-guided resource allocation processes to cope with increasing returns, has since been a topic of continuing interest. Ramsey's contribution was enshrined as a durable piece with a resurgence of

* Research support from the National Science Foundation is gratefully acknowledged. The research of the first author was also supported by the Warshow endowment at Cornell and that of the second author by an Alfred P. Sloan Research fellowship. We would also like to thank Professors R. N. Bhattacharya, C. Clark, J. Chipman and L. Hurwicz for helpful suggestions.

interest in intertemporal economics in the fifties. But neither John Keynes, the editor of the *Economic Journal* who was most appreciative of Ramsey's talents, neither the subsequent writers on 'growth theory' in Cambridge, England (nor, for that matter, those in Cambridge, Massachusetts), have made any precise suggestion towards incorporating increasing returns in a Ramsey-type exercise.

A concise discussion about a phase of increasing returns in a production process appeared in Frank Knight's Ph.D. thesis at Cornell (subsequently published as *Risk, Uncertainty and Profit* (1921), see pp.100–1), and casual references to increasing returns in the context of capital accumulation were, of course, made from time to time. Joan Robinson (1956, chapter 33) and John Hicks (1960) alluded to the importance of Young's ideas, but somehow, increasing returns had to be discussed by crossing 'the boundary of topics that can usefully be discussed in the framework of ... simplifying assumptions' (Robinson (1956), p.336). Systematic and formal expositions of increasing returns in a Ramsey-type model of dynamic optimization emerged much later, and, mostly in the context of relatively recent 'pragmatic' concern in the economics of exhaustible and natural resources. The paper by Clark (1971) in *Mathematical Biosciences* is indeed a landmark in this area, and our primary objective is to bring together a collection of subsequent results in a fairly simple framework of dynamic optimization under uncertainty. To be sure, a *deterministic* exercise can be viewed as a very special case of our formal stochastic model, and our development of ideas and exposition owe much to the insights gained from such deterministic exercises following up Clark's paper. As with Frank Knight, our object is refinement (not reconstruction), rigor – not realism.

1.1 Recent Literature: An Overview

Discrete time deterministic models of dynamic optimization with an 'S-shaped production function' have been explored by Majumdar and Mitra in (1982) and (1983). These papers dealt with three different evaluation criteria: intertemporal efficiency, maximization of a discounted sum of one-period felicities (linear felicities in (1983)), and the case of undiscounted optimality in the sense of 'overtaking'. The linear case had been studied earlier by Clark (1971). Dechert and Nishimura (1983) made significant improvements of results obtainable in the discounted case, and further extensions in different directions were made by Majumdar and Nermuth (1982), and Mitra and Ray (1984).

One-good models of dynamic optimization under uncertainty in the classical 'convex' environments were studied by Brock and Mirman (1972) (strictly concave felicities), Jaquette (1972) and Reed (1974) (linear felicities, multiplicative shocks). A useful source of related references is Mirman and Spulber (1982).

1.2 A Reader's Guide

We develop the formal model of sequential decision making under uncertainty as a special case of the stochastic discounted dynamic programming model explored by Blackwell (1965) and others. The set of technologically feasible 'plans' or policies is non-convex, but the evaluation criterion is a discounted sum of expected (concave) felicities. In Section 2, we summarize some basic theorems on the existence of optimal stationary policies. These are obtained for both linear and strictly concave felicities. The value function is shown to be continuous and to satisfy the functional equation of dynamic programming (see (8) and (9)). In Section 3, the main result (Theorem 2) is that any optimal investment policy function $h(y)$ is monotonically non-decreasing in the stock y. Such a policy function is a selection from a correspondence, and is, in general, not unique. Without the assumption of convexity of the feasible set, the optimal consumption policy function $c(y)$ is not necessarily monotone non-decreasing in y. In fact, under some additional assumptions, we prove that $c(y)$ is non-decreasing if and only if the value function V is concave. In Section 4, we identify conditions under which optimal policies are interior and can be characterized (in the case of strictly concave felicities) by a stochastic version of the Ramsey–Euler condition (see (13)). We should emphasize that we could prove the interiority property only under the assumption that utility of zero consumption is minus infinity. In the context of convex models, an 'Inada condition' on the boundary (namely, that marginal utility goes to infinity as consumption drops to zero) is sufficient to guarantee that optimal input, consumption and stock processes are interior (see Theorem 19). Without convexity, however, we were able to assert only that input and stock (but not necessarily consumption) processes are interior. Theorem 18 and Example 3 clarify this issue. Section 5 deals with the problem of non-uniqueness of optimal processes in non-convex models. The value function is differentiable at an initial stock if and only if there is a unique optimal process from the same stock, and non-differentiability can only arise on a set that is at most countable. In Section 6, some results on the convergence of optimal inputs are obtained (see Theorems 11–13), and the behavior of optimal inputs in non-convex models is contrasted with that in convex models.

In convex stochastic models, the 'turnpike' property obtains: regardless of the initial stock, the optimal input process converges in distribution to a unique invariant distribution. With non-convexity, there are potentially many invariant distributions and, furthermore, the one to which an optimal input process converges depends on the initial stock (the turnpike one takes depends on the point of departure!). The analysis of the optimal process that leads to Theorems 11 and 12 depends, however, on the assumptions that there are finitely many random events at any date, and (more importantly) different 'splitting' condition ensuring that there is sufficient variability in production yields a strong turnpike result (Theorem 13). Section 7 contains additional incidental results and provides a technical overview of the related literature. All the proofs are relegated to Section 8.

1.3 The Role of Non-convexity

What are the most striking differences in the qualitative properties of optimal decisions when non-convexities are introduced? Ignoring uncertainty for the moment, let us stress a couple of important aspects (see Majumdar and Mitra (1982) for details). In a deterministic model with an S-shaped production function, it turns out that *average productivity* along a feasible program has a crucial role in signalling intertemporal inefficiency in contrast with the classical model where the remarkable Cass condition can be precisely expressed solely in terms of marginal productivities. Secondly, when future utilities are discounted, the qualitative properties of optimal programs depend critically on the magnitude of the discount factor. Roughly speaking, when discounting is 'mild', optimal programs behave as in the undiscounted case (converging to a unique optimal stationary program), when discounting is 'heavy', optimal programs approach extinction. Also, in the 'intermediate' range of discounting, the long-run behavior depends critically on the initial stock. Contrast this with the 'classical' turnpike literature where the long-run behavior of optimal programs is invariant with respect to the initial stock.

The fact that solutions to dynamic optimization problems in one-good models display striking monotonicity properties has been an important by-product of research efforts in the area reviewed in the present paper. As noted above, a version of monotonicity continues to hold even when uncertainty is introduced, and such properties have been exploited to study the dynamic behavior of optimal decisions. The main shortcoming of the present paper, however, is an inadequate analysis of the nature of optimal processes under uncertainty with an S-shaped production function introduced by Frank Knight. (The Inada condition (T.8) in Section 6.1 is *not* consistent with an S-shape.) A deeper understanding of this case is certainly desirable, and – to us – is the most important gap in the literature on stochastic dynamic optimization with non-convexities.

2 THE MODEL

2.1 Sequential Decisions

The standard framework of stochastic dynamic programming (see, e.g., resource allocation under uncertainty. In each period t, the planner observes the current stock y_t of a particular good, and chooses 'an action': some point a in $A \equiv [0,1]$. One interprets a as the fraction of y_t to be used as input in period t, and refers to $x_t \equiv ay_t$ as the *input* in period t. As a result of the decision on a, the stock in the next period $t+1$ is determined according to the following relationship:

$$y_{t+1} = f(x_t, r_{t+1}) \equiv f(ay_t, r_{t+1}) \tag{1}$$

where f is the gross output function (satisfying the assumptions introduced below), and (r_t) is a sequence of independent, identically distributed random variables ('shocks' to the production process). Choice of a also determines the consumption $c_t \equiv (1-a)y_t$. Consumption generates an immediate return or utility according to a function u (satisfying the assumptions listed below). In other words, choice of a generates utility defined as

$$u(c_t) \equiv u((1-a)y_t)$$

Note that the decision on a is made *before* the realization of r_{t+1}; in the next period, y_{t+1} is observed after the realization of r_{t+1} and the same choice problem is repeated.

A *policy* $\underline{\pi}$ is a sequence $\underline{\pi} = (\pi_t)$ where π_t specifies the action in the t-th period as a function of the previous history $\eta_t = (y_0, a_1, \ldots, a_{t-1}, y_t)$ of the system, by associating with each η_t (Borel measurably) an element a of A (hence, the input $x_t^{(\pi)} \equiv ay_t \equiv \pi_t(\eta_t)y_t$ and the consumption $c_t^{(\pi)} \equiv (1-a)y_t \equiv [1-\pi_t(\eta_t)]y_t$). Any Borel function g: $R_+ \to A$ defines a policy: whenever $y \geqq 0$ is observed, choose $a = g(y)$ irrespective of when and how the stock y is attained. The corresponding policy $\underline{\pi} = (g^{(\infty)})$ is a stationary policy and g is the policy function generating the stationary policy.

A policy $\underline{\pi}$ associates with each initial stock y a corresponding t-th period expected utility $Eu(c_t(\underline{\pi}))$ and an expected discounted total utility defined as

$$V_{\underline{\pi}}(y) = \sum_{t=0}^{\infty} \delta^t Eu(c_t(\underline{\pi})), \tag{2}$$

where δ is the discount factor, $0 < \delta < 1$. The measure theoretic foundation underlying the expectation operation in (2) is fully spelled out in Blackwell (1965).

A policy $\underline{\pi}^* = (\pi_t^*)$ is optimal if $V_{\underline{\pi}^*}(y) \geqq V_{\underline{\pi}}(y)$ for all $y > 0$ and all policies $\underline{\pi}$. We call $V_{\underline{\pi}^*}$ the optimal) value function defined by $\underline{\pi}^*$. Note that if $\underline{\pi}^*$ and $\bar{\underline{\pi}}$ are both optimal policies, $V_{\underline{\pi}^*} = V_{\bar{\underline{\pi}}}$. Hence, we shall often drop the subscript in referring to the value function V.

2.2 Environment, Technology and Utility

Let $\mathcal{E}$ be a compact set of positive real numbers. Two examples of $\mathcal{E}$ are of particular interest: (a) $\mathcal{E}$ is a finite set; (b) $\mathcal{E}$ is a closed interval $[b_1, b_2]$ in positive reals. The elements of $\mathcal{E}$ are alternative states of the environment in any period. Let (r_t) be a sequence of independent, identically distributed random variables with values in $\mathcal{E}$ and a common distribution γ.

The technology is described by a gross output function f: $R_+ \times \mathcal{E} \to R_+$ satisfying the following:

(T.1) There is $\beta > 0$ such that for all $x \geqq \beta$, $x > f(x,r)$ for all $r \in \mathcal{E}$.

(T.2) For each $r \in \mathcal{E}$, $f(\cdot,r)$ is continuous on R_+.

(T.3′) For each $r \in \mathcal{E}$, $f(\cdot,r)$ is non-decreasing on R_+.

A stronger version of (T.3′) is

(T.3) For each $r \in \mathcal{E}$, $f(\cdot,r)$ is strictly increasing on R_+.

The immediate return or utility function u: $R_+ \to R$ is assumed to satisfy

(U.1) u is continuous on R_+.

(U.2) u is strictly increasing on R_+.

(U.3) u is strictly concave on R_+.

We shall point out that most of the important results can also be proved if instead of (U.1) we have

(U.1′) u is continuous on R_{++} and $\lim_{c \downarrow 0} u(c) = -\infty$.

Under (U.1′), to ensure that the value function is finite, i.e. $V(y) > -\infty$, we impose the following assumption:

(T.4) There is a $k > 0$ such that for all $0 < x < k$, $f(x,r) > x$ for each r.

When we want to accommodate a linear utility function we replace (U.3) by:

(U.3′) u is concave on R_+.

$\underset{\sim}{x}^{\pi}=(x_t^{(\pi_t)})$, a consumption process $\underset{\sim}{c}^{\pi}=(c_t^{(\pi_t)})$ and a *stock* process $\underset{\sim}{y}^{\pi}=(y_t^{(\pi_t)})$ according to the description in Section 2.1. Formally,

$$x_0\equiv\pi_0(y)y,\ c_0\equiv[1-\pi_0(y)]y,\ y_0=y \tag{3}$$

and, for each $t\geqq 1$, and for each partial history $\eta_t\equiv(y_0,a_0,\ldots,y_{t-1},a_{t-1},y_t)$ one has:

$$x_t(\eta_t)\equiv\pi_t(\eta_t)y_t,\ c_t\equiv[1-\pi_t(\eta_t)]y_t,\ y_t\equiv f(x_{t-1},r_t) \tag{4}$$

Clearly,

$$c_0+x_0=y_0;$$

$$c_t+x_t=f(x_{t-1},r_t)\text{ for all }t\geqq 1; \tag{5}$$

$$c_t\geqq 0,\ x_t\geqq 0\text{ for all }t\geqq 0.$$

For brevity, we call $(\underset{\sim}{x}^{\pi^*},\underset{\sim}{c}^{\pi^*},\underset{\sim}{y}^{\pi^*})$ an optimal (resp. input, consumption, stock) process generated by an optimal policy $\underset{\sim}{\pi}^*$ (when it exists).

We first note a useful boundedness property in our model.

Lemma 1: Assume (T.1), (T.2) and (T.3′), and let $y>0$ be any initial stock. If $\underset{\sim}{\pi}$ is any policy generating the stock process $\underset{\sim}{y}^{\pi}$, then for all $t\geqq 0$

$$0\leqq y_t\leqq\max(\beta,y) \tag{6}$$

Proof: An induction argument on t is easy to construct.

QED

It follows from (5) and (6) that the processes $\underset{\sim}{x}^{\pi}$ and $\underset{\sim}{c}^{\pi}$ generated by $\underset{\sim}{\pi}$ also satisfy for all $t\geqq 0$

$$x_t\leqq\max(\beta,y),\ c_t\leqq\max(\beta,y) \tag{7}$$

In what follows we restrict the initial stock $y\leqq\beta$. Define $S\equiv[0,\beta]$ and denote by $q(\cdot|y,a)$ the conditional distribution of y_{t+1} given the stock y and action a in period t [as determined by f and the common distribution γ of r_{t+1}]. One can verify:

Lemma 2: Under (T.2), if a sequence $(y^n,a^n)\in S\times A$ converges to $(y,a)\in S\times A$, the sequence $q(\cdot|y^n,a^n)$ converges weakly to $q(\cdot|y,a)$.

2.3 Existence of an Optimal Stationary Policy

The basic existence theorem of Maitra (1968) when applied to our case leads to the following:

Theorem 1: Assume (T.1), (T.2), (T.3′) and (U.1). There exists an optimal stationary policy $\underset{\sim}{\pi}^* = (\hat{h}^{(\infty)})$ where $\hat{h}$: $S \to A$ is a Borel measurable function. The value function $V_{\underset{\sim}{\pi}^*}$ defined by $\underset{\sim}{\pi}^*$ on S is continuous and satisfies

$$V_{\underset{\sim}{\pi}^*}(y) = \max_{a \in A} [u((1-a)y) + \delta \int V_{\underset{\sim}{\pi}^*}[f(ay, r)]d\gamma] \tag{8}$$

$$= u[y - \hat{h}(y) \cdot y] + \delta \int V_{\underset{\sim}{\pi}^*}[f(\hat{h}(y)y, r)]d\gamma \tag{9}$$

Remarks

(1) The existence of $\underset{\sim}{\pi}^*$ requires assumptions weaker than those listed above. The stronger continuity conditions are used to establish the continuity of $V_{\underset{\sim}{\pi}^*}$ by readily adapting the proof in Maitra (1968).

(2) If (U.1) is replaced by (U.1′) one appeals to Schäl (1975) to establish Theorem 1.

(3) We refer to the function $h(y) \equiv \hat{h}(y) \cdot y$ as an optimal investment policy function and the function $c(y) \equiv [1 - \hat{h}(y)]y$ as an optimal consumption policy function.

To simplify notation, we write $(\underset{\sim}{x}^*, \underset{\sim}{c}^*, \underset{\sim}{y}^*)$ to denote the optimal input, consumption and stock processes generated by $\underset{\sim}{\pi}^*$ (see (4) and (5)).

3 MONOTONICITY OF OPTIMAL INVESTMENT POLICY FUNCTION

3.1 Strictly Concave Utility Function

We first establish a (weak) monotonicity property of optimal investment policy functions. In all of this section, (U.1) can be replaced with (U.1′) and (T.4).

Theorem 2: Assume (T.1)–(T.3) and (U.1)–(U.3). Then if h is an optimal investment policy function, h is non-decreasing, i.e. '$y > y'$' implies '$h(y) \geqq h(y')$'.

Remarks

(1) Strong monotonicity ['$y>y'$' implies '$h(y)>h(y')$'] of h is proved later when the optimal process is characterized by the stochastic Ramsey–Euler conditions.

(2) The proof of Theorem 2 relies critically on *strict* concavity of the utility function u.

(3) It should be emphasized that Theorem 2 is concerned with the monotonicity of the optimal investment policy function $h(y)\equiv\hat{h}(y)\cdot y$ where the existence of $\hat{h}$ is proved in Theorem 1. However, $\hat{h}$ is a selection from a correspondence and is, in general, not unique. (Uniqueness of $\hat{h}$ leads to the continuity of $\hat{h}$ (hence h) and is obtainable in the classical 'convex' model studied by Brock–Mirman–Zilcha.)

(4) The strategy of proof that we follow in Theorem 2 is due to Dechert and Nishimura (1983), who established a parallel result in their deterministic model with a non-convex technology.

(5) In Theorem 2 above we may replace (T.3) with (T.3′).

The functional equation characterizing the value function V can be recast as (see (8)):

$$V(y)=\max_{0\leqslant x\leqslant y}[u(y-x)+\delta\int V[f(x,r)]d\gamma] \tag{10}$$

$$=u(c(y))+\delta\int V[f(h(y),r)]d\gamma \tag{11}$$

Given $y>0$, define $\varphi(y)$ to be the set of all values x where the right side of (10) attains its maximum. The continuity properties in our model imply that $\varphi(y)$ is non-empty, and that $h(y)$ is a selection from the correspondence φ.

It is important to note that a stronger version of Theorem 2 is true. If $\{x_t\}_0^\infty$, $\{x_t'\}_0^\infty$ are optimal input processes from y,y' then $y>y'$ implies $x_0\geqq x_0'$. This implies the following ordering relation on φ. If A and B are two subsets of R_+, then we say $A\geqq B$ if $a\in A$ and $b\in B$ implies $a\geqq b$. φ then has the property that $y>y'$ implies $\varphi(y)\geqq\varphi(y')$.

Define

$$\underline{h}(y)=\min[x\colon x\in\varphi(y)]$$

$$\bar{h}(y)=\max[x\colon x\in\varphi(y)]. \tag{12}$$

3.2 Concave Utility Function

In this subsection, we assume (T.1)–(T.3), (U.1), (U.2) and (U.3′). It should be stressed that the strict concavity assumption (U.3) on u is replaced by the less restrictive concavity assumption (U.3′). Our first result provides a characterization of the correspondence φ and its selections $\underline{h}$ and $\bar{h}$.

Theorem 3: Under (T.1), (T.2), (T.3′), (U.1), (U.2) and (U.3′):

(a) φ is an upper semicontinuous correspondence;

(b) $\underline{h}$ is well defined, left continuous and non-decreasing;

(c) $\bar{h}$ is well defined, right continuous and non-decreasing.

Corollary 3: There is a Borel selection $\hat{h}$: $S \rightarrow A$ such that the corresponding optimal investment policy function $h(y) \equiv \hat{h}(y) \cdot y$ is non-decreasing and right continuous.

Remark

There is also a selection such that h can be made to satisfy left continuity (and weak monotonicity).

4 INTERIOR OPTIMAL PROCESSES AND THE STOCHASTIC RAMSEY–EULER CONDITION

We now impose (U.1′) and (T.4) and derive the property that the optimal processes are interior (a.s.). In addition to (T.1)–(T.4) and (U.1′)–(U.3), we make the following assumptions in this section:

(T.5) $f(x,r) = 0$ if and only if $x = 0$.

(T.6) $f(x,r)$ is continuous on $R_+ \times \mathcal{E}$.

Theorem 4: (Interiority) Under (T.1)–(T.6) and (U.1′)–(U.3), if $(\underline{x}^*, \underline{c}^*, \underline{y}^*)$ is an optimal process from some initial stock $y > 0$, then for all $t \geqq 0$, $x_t^*(\eta_t) > 0$, $c_t^*(\eta_t) > 0$ and $y_t^*(\eta_t) > 0$ for all η_t.

In studying the dynamic behavior of optimal processes, it is useful to exploit the stochastic Ramsey–Euler conditions characterizing an optimal process. We now introduce additional assumptions:

(U.4) $u(c)$ is continuously differentiable at $c > 0$.

Theorem 5: (Stochastic Ramsey–Euler Condition) Under (T.1)–(T.7) and (U.1′)–(U.4) if h is an optimal investment policy function and $c(y) = y - h(y)$,

$$u'(c(y)) = \delta \int u'(c(f(h(y),r)))f'(h(y),r)d\gamma \tag{13}$$

Corollary 5: If (13) holds, h: $R_+ \to R_+$ is strictly increasing.

So far we have discussed monotonicity of the optimal investment policy function. The following result throws light on monotonicity of the optimal consumption policy function.

Theorem 6: If the value function V is concave, then it is differentiable at each $y > 0$. Furthermore, $c(y)$ is non-decreasing if and only if V is concave.

5 SOME PROPERTIES OF THE VALUE FUNCTION AND UNIQUENESS OF OPTIMAL PROCESSES

We now establish some differentiability properties of the value function and relate these to the question of uniqueness of optimal processes. Recall that in the classical 'convex' model, with the assumption (U.3), there is a unique optimal process. With a non-convex technology, uniqueness cannot be asserted even in deterministic models. However, the nature of non-uniqueness can be clarified by examining the value function V. In this section we assume that (U.1′), (U.2)–(U.4) and (T.1)–(T.7) all hold.

Going back to (10), recall that the correspondence $\varphi(y)$ is not in general single valued, and an optimal investment policy function is a selection from $\varphi(y)$.

Lemma 3: There is a countable set D in S, such that if y is not in D, $\varphi(y)$ is single valued.

Remark: Recall that φ is upper semicontinuous as a correspondence, hence if $\varphi(y)$ is single valued, continuity of $h(y)$ and $c(y)$ follows.

Lemma 4: The left- and right-hand derivative (denoted by V^-, V^+, respectively) of V exist at all $y > 0$; $V^-(y) \leqq V^+(y)$. Furthermore, $\{y\colon V^-(y) < V^+(y)\}$ is at most countable.

The following theorem throws light on non-uniqueness of optimal processes:

then $x_t = x'_t$ and $c_t = c'_t$ for all $t \geqq 1$. Furthermore, V is differentiable at some $y > 0$ if, and only if there is a unique optimal process from y.

Remark: The results reported above are stochastic versions of the parallel deterministic results in Dechert and Nishimura (1983).

Finally, we come to the useful 'envelope theorem':

Theorem 8: Suppose that V is differentiable at some $y > 0$. Then

$$V'(y) = u'(c(y)) = \delta \int V'[f(h(y),r)]f'(h(y),r)d\gamma \tag{14}$$

6 DYNAMIC BEHAVIOR OF AN OPTIMAL INPUT PROCESS

We now focus on the stochastic process

$$x_{t+1} = h[f(x_t, r_{t+1})] \tag{15}$$

where h is an optimal investment policy function. Two cases are considered. First, we assume that $\mathcal{E}$ is finite and that the production function satisfies the Inada condition at the origin ('infinite derivative at zero'). One can allow for a phase of increasing returns once a specific positive level of input has been committed. The dynamic behavior of the process (15) turns out to be quite different from that in the 'classical' case of a strictly concave f. Next, we impose a condition that requires that there be sufficient variability in the production function. We then conclude that the distribution function of x_t converges uniformly to the distribution function of a unique invariant distribution, regardless of the level of initial stock.

In both cases we shall use a result of Dubins and Freedman (1966); to state this result we require the following notation.

Recall that γ is the probability distribution of r on $\mathcal{E}$. Let $\gamma^n = \gamma x ... x\gamma$ (n-times) be the product measure induced by γ on $\mathcal{E}^n$. Let $r^n = (r_1, ..., r_n)$ be a generic element of $\mathcal{E}^n$ and define for any x in S

$$H(x,r) = h[f(x,r)]$$

$$H^n(x,r^n) = H(H(...(H(x,r_1),r_2),...,r_n) \tag{16}$$

If μ is any probability on S, define the probability $\gamma^n\mu$ on S by the relation

$$\gamma^n\mu(A) = \int_S \gamma^n(\{r^n \in \mathcal{E}^n | H^n(x,r^n) \in A\})\mu(dx) \tag{17}$$

Let S' be a closed interval in S. S' is said to be γ-invariant if $\gamma(\{r \text{ in } \mathcal{E} | H(x,r) \in S' \text{ for all } x \text{ in } S'\}) = 1$. $\bar{x}$ in S is a γ-fixed point if the singleton set $\{\bar{x}\}$ is γ-invariant.

Let S' be γ-invariant. For any $n = 1,2,\ldots$, the probability γ^n is said to split on S' if there is a z in S' such that

$$\gamma^n(\{r^n \text{ in } \mathcal{E}^n | H^n(x,r^n) \leqq z \text{ for each } x \text{ in } S'\}) > 0 \tag{18}$$

$$\gamma^n(\{r^n \text{ in } \mathcal{E}^n | H^n(x,r^n) \geqq z \text{ for each } x \text{ in } S'\}) > 0$$

A probability μ on S is said to be an invariant probability on S' if the support of μ is a subset of S', and for any Borel set A in S,

$$\gamma\mu(A) = \mu(A) \tag{19}$$

An invariant distribution is the distribution function of an invariant probability. We may now state a modification of Dubins and Freedman (1966, Corollary 5.5, p.842):

Theorem 9: Suppose for some γ-invariant closed interval S', and for some integer n, γ^n splits on S'. Suppose further that there are no γ-fixed points $\bar{x}$ in S', and also that the function $H(\cdot,r)$ is monotone non-decreasing on S', for γ a.e. r in $\mathcal{E}$. Then there is one and only one invariant probability μ on S'; and for each probability ν whose support is a subset of S', the distribution function of $\gamma^n\nu$ converges uniformly to the distribution function of μ.

6.1 Optimal Input Process When $\mathcal{E}$ Is Finite and the Inada Condition Holds

In this subsection, in addition to (T.1)–(T.7) and (U.1′)–(U.4), we assume that the following conditions hold:

(E.1) $\mathcal{E}$ is finite.

(T.8) For each $r \in \mathcal{E}$, $\lim_{x \downarrow 0} f'(x,r) = \infty$ (Inada condition at the origin).

(T.9) For any fixed $x > 0$, there does not exist any $\bar{y}$ in S such that

$$\gamma(\{r \in \mathcal{E} | f(x,r) = \bar{y}\}) = 1.$$

Since, from Corollary 5, h is strictly increasing, (T.9) implies there are no positive γ-fixed points in S. Roughly speaking, we show that the optimal process (14) enters one of a number of disjoint sets and stays in it. Unlike the

Brock–Mirman–Zilcha case of strictly concave production functions (in which x_t converges in distribution to an invariant distribution irrespective of the initial stock), the set into which x_t eventually enters depends very much on the starting point.

Some new notations are (alas!) needed. Let

$$f_m(x) \equiv \min_r f(x,r) \text{ and } f_M(x) \equiv \max_r f(x,r).$$

Clearly, f_m and f_M are well-defined continuous functions on R_+. Recall $H(x,r) \equiv h[f(x,r)]$. Define

$$\bar{H}(x,r) \equiv \bar{h}[f(x,r)] \tag{20}$$

$$\underline{H}(x,r) \equiv \underline{h}[f(x,r)]$$

where the functions $\bar{h}$ and $\underline{h}$ are defined in (12). Finally, let

$$H_m(x) = \min_r H(x,r) \text{ and } H_M(x) = \max H(x,r).$$

$\bar{H}_m, \bar{H}_M$ and $\underline{H}_m, \underline{H}_M$ may be similarly defined by replacing H by $\bar{H}$ and $\underline{H}$. From the monotonicity of f and h, one can show that (we prove this in Section 8):

$$H_m(x) = h(f_m(x)),\ H_M(x) = h(f_M(x)) \tag{21}$$

The main results on the long-run behavior of x_t rely on an analysis of fixed points of the maps just introduced. Fortunately, these fixed points are independent of the choice of optimal policy function. This is stated formally below.

Lemma 5

(a) H, $\bar{H}$, $\underline{H}$ all have the same fixed points;

(b) H_M, $\bar{H}_M$, $\underline{H}_M$ all have the same fixed points;

(c) H_m, $\bar{H}_m$, $\underline{H}_m$ all have the same fixed points.

The following lemma is important for the results that follow.

Lemma 6: There exists $\varepsilon > 0$ such that $H(x,r) > x$ for all x in $(0,\varepsilon)$ and all r in $\mathcal{E}$.

Next, let us define

$$y_m = \min\{x > 0:\ H_m(x) = x\},\ y_M = \max x > 0:\ H_M(x) = x\} \tag{22}$$

It should be stressed that Lemma 5 implies that the numbers x_m, y_m, x_M, y_M in (22) are independent of the selection of h. The following facts are gathered in the form of a lemma:

Lemma 7

(a) The points x_m, y_m, x_M, y_M are well defined;

(b) $y_m > 0$;

(c) $H_m(x) > x$ for all x in $(0, y_m)$, $H_m(x) < x$ for all x in (x_m, ∞);

(d) $H_M(x) > x$ for all x in $(0, x_M)$ and $H_M(x) < x$ for all x in (y_M, ∞);

(e) $y_m \leqq x_M$ and $x_m \leqq y_M$.

From Lemma 7, there are two possible configurations:

(A) $x_m \leqq x_M$ and (B) $x_m > x_M$ (see Fig. 19.1 below).

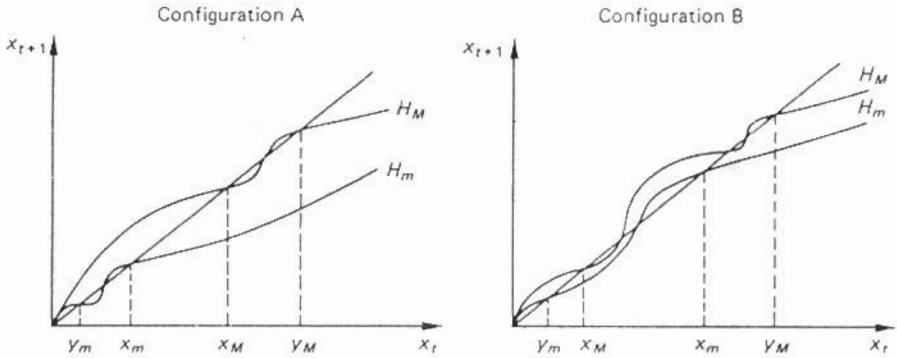

Figure 19.1

Configuration A holds when for each r, $f(\cdot,r)$ is strictly concave. (See Theorem 17 in Section 7.1.) One can, however, give examples of non-convex technologies where configuration A still holds.

The dynamic behavior of the optimal input process under configurations A and B is stated in Theorems 11 and 12 below. In both cases we exploit Theorem 9 via another theorem of Dubins and Freedman (1966), which we state below:

Theorem 10: Let S' be a γ-invariant closed interval in S. Suppose that for γ a.e. r in $\mathcal{E}$, $H(\cdot,r)$ is continuous and monotone non-decreasing on S', and there are no γ-fixed points in S'. If there is a unique minimal γ-invariant

Theorem 9 hold. The hypothesis that $H(\cdot,r)$ be continuous on S' for γ a.e. r in $\mathcal{E}$, may be dropped if instead we assume that $\mathcal{E}$ is finite and Lemma 5(a) holds.

We now state two main results characterizing the asymptotic properties of the optimal input process, $\{x_t\}$. Let $F_t(x)$ be the distribution function of x_t, i.e. $F_t(x) = \text{Prob}(\{x_t \leqslant x\})$.

Theorem 11: If configuration A holds, $F_t(x)$ converges as $t \to \infty$ uniformly in x to a unique invariant distribution $F(x)$, independently of initial stock y_0. The support of F is $[x_m, x_M]$.

Theorem 12: Suppose configuration B holds. If $x_0 \in (0, x_M]$ (resp. $x_0 \in [x_m, \infty)$) then $F_t(x)$ converges as $t \to \infty$ uniformly in x to the invariant distribution $\bar{F}(x)$ (resp. $F(x)$) whose support is a subset of $[y_m, x_M]$ (resp. $[x_m, y_M]$). $\bar{F}(x)$ (resp. $F(x)$) is the unique invariant distribution with support a subset of $(0, x_M]$ (resp. $[x_m, \infty)$).

6.2 Case When Production Is 'Very Stochastic'

In this subsection we assume (T.1)–(T.3) and (U.1)–(U.3) (enough to ensure the monotonicity result of Theorem 2 holds). We next impose (T.10) below, which requires that there be sufficient variability in production:

(T.10) There is $z > 0$ in S such that

$$\gamma(\{r \in \mathcal{E} \mid f(x,r) \leqq z \text{ for each } x \in S\}) > 0$$

and

$$\gamma(\{r \in \mathcal{E} \mid f(x,r) \geqq z \text{ for each } x \in S\}) > 0$$

Recall that $F_t(x)$ is the distribution function of x_t.

Theorem 13: $F_t(x)$ converges as $t \to \infty$ uniformly in x to a unique invariant distribution $F(x)$, independently of the initial stock.

Example: Suppose $f(x,r) = 1/\{1 + me^{-kx}\} + r$, with $k > 0$ and $m > 0$. This is the logistic growth function with additive shocks. If $\mathcal{E} = [a,b]$ is the support of γ, then $b - a > m/1 + m$ will ensure that (T.10) holds and we obtain the conclusions of Theorem 13.

7 SOME ADDITIONAL RESULTS

7.1 The Convex Environment: An Overview

We now indicate how the standard results that hold in the 'convex environment' (i.e. with a strictly concave production function) can be obtained as special cases of our earlier results. Suppose that we impose the following concavity assumption:

(T.11) For each fixed $r \in \mathcal{E}$, $f(\cdot, r)$ is strictly concave on S.

The following results summarize some uniqueness and concavity properties when (T.11) is added to our model:

Theorem 14: Under (T.1)–(T.3), (T.11), (U.1), (U.2) and (U.3):

(a) If $(\underset{\sim}{x}, \underset{\sim}{c}, \underset{\sim}{y})$ and $(\underset{\sim}{x}', \underset{\sim}{c}' \underset{\sim}{y}')$ are optimal (resp. input, consumption and stock) processes from the initial stock $y > 0$, then for each $t \geqq 0$, $x_t = x'_t$, $c_t = c'_t$ and $y_t = y'_t$ a.s.

(b) $\varphi(y)$ is single valued at each $y > 0$.

(c) The value function $V(y)$ is concave in y.

(d) If, in addition, (U.3) holds, the value function, $V(y)$, is strictly concave in y.

Remarks

(1) In Theorem 14 above, (U.1) may be replaced with (U.1′) and (T.4).

(2) Without (U.3), the strict concavity of u, Theorem 14(d) may not hold.

From Theorem 2 we obtained the monotonicity of the optimal investment policy function, $h(y)$. In general, the monotonicity of the optimal consumption policy function, $c(y)$, cannot be asserted for non-convex technologies. Further, the functions $h(y)$ and $c(y)$ are not necessarily continuous.

We indicate below, however, that in the special case where (T.11) holds, such monotonicity and continuity results may be obtained.

Theorem 15: Under (T.1)–(T.3), (T.11), (U.1), (U.2) and (U.3′):

(a) $h(y)$ is continuous and non-decreasing in y;

(b) $c(y)$ is continuous and non-decreasing in y.

Remarks

(1) Again we note that in Theorem 15, (U.1) may be replaced with (U.1′) and (T.4).

(2) Theorem 15 uses the weaker assumption (U.3′), hence allows for linear utility functions.

Next, we obtain the differentiability of the value function and the 'envelope theorem'.

Theorem 16: Under (T.1)–(T.7), (T.11) and (U.1′), (U.2)–(U.4):

(a) The value function, $V(y)$, is differentiable at each $y > 0$;

(b) The following 'envelope theorem' holds at each $y > 0$;

$$V'(y) = u'(c(y)) = \delta \int V'(f(h(y),r)] f'(h(y),r) d\gamma \tag{23}$$

The following corollary strengthens Theorem 15.2

Corollary 16: Under the hypotheses of Theorem 16.3:

(a) $h(y)$ is continuous and strictly increasing in y;

(b) $c(y)$ is continuous and strictly increasing in y.

Next, recall that in Section 6.1 we obtained the dynamic behavior of optimal input processes. We indicated that this behavior depends upon whether configuration A or B holds. We now show that under (T.11), configuration A holds and we obtain the convergence of the optimal input process to a unique invariant distribution irrespective of where the process begins.

Theorem 17: Assume (T.1)–(T.9), (T.11), (U.1′), (U.2)–(U.4) and (E.1). Then configuration A, hence, the conclusions of Theorem 11 hold.

Remarks

(1) The proof of the result above that configuration A holds under (T.11) is due to Mirman and Zilcha (1975, lemma 2).

(2) In Theorems 16 and 17 the assumptions (U.1′) and (T.4) are used. This is because (U.1′) and (T.4) are required to show that optimal processes are interior (Theorem 4); this is then used to prove the stochastic Ramsey–Euler condition (Theorem 5), which is crucial to the proofs of Theorems 16 and 17. Consider the following Inada condition at the origin:

(U.5) $\lim_{c \downarrow 0} u'(c) = \infty$

In Theorem 19 of Section 7.2 below, we indicate that Theorem 4 (Interior Optimal Processes), continues to hold if (U.1′) and (T.4) are replaced with (U.1), (U.5) and (T.11). Hence, in Theorems 16 and 17, we may replace (U.1′) and (T.4) with (U.1) and (U.5).

7.2 Interior Optimal Processes

In this subsection we discuss Theorem 4 (Interior Optimal Processes) when the assumptions (U.1′) and (T.4) are replaced with (U.1) and (U.5) (the 'Inada condition' at the origin), which we repeat here:

(U.5) $\lim_{c \downarrow 0} u'(c) = \infty$.

We will also require,

(T.12) $\liminf_{x \downarrow 0} f'(x,r) > 0$ for each r.

First we indicate that if (U.1′) and (T.4) are replaced with (U.1), (U.5) and (T.12) in Theorem 4, we can show that the optimal input and stock processes are interior.

Theorem 18: Under (T.1)–(T.3), (T.5)–(T.7), (T.12), (U.1) and (U.2)–(U.5), if $(\underset{\sim}{x}^*, \underset{\sim}{c}^*, \underset{\sim}{y}')$ is an optimal process from some initial stock $y > 0$, then for all $t \geqq 0$, $x_t^*(\eta_t) > 0$ and $y_t^*(\eta_t) > 0$ for all histories η_t.

Remarks

(1) Under the hypotheses of Theorem 18, we are unable to show that optimal consumption processes are interior.

(2) Theorem 18 may fail if (T.12) does not hold (see Section 7.3, Example 3).

(3) Notice that (T.11) is not required in Theorem 18.

We now indicate that under (T.11), Theorem 18 may be strengthened to show that optimal consumption processes (as well as input and stock processes) are interior.

Theorem 19: Under (T.1)–(T.3), (T.5)–(T.7), (T.11), (U.1) and (U.2)–(U.5), if $(\underset{\sim}{x}^*, \underset{\sim}{c}^*, \underset{\sim}{y})$ is an optimal process from some initial stock $y > 0$, then for all $t \geqq 0$, $x_t^*(\eta_t) > 0$, $c_t^*(\eta_t) > 0$ and $y_t^*(\eta_t) > 0$ for all histories η_t.

7.3 Some Examples

We shall now present a collection of examples where one or more of the assumptions in our model do not hold, and as a result, some conclusions that we have derived fail to remain valid.

Example 1: We relax the assumption (U.3) in Theorem 2 and show that with a linear utility function, an optimal investment policy function need not be monotone non-decreasing.

Let $u(c)=1/\delta c$ for $c \geqslant 0$ and $f(x,r)=1/\delta x$ if $0 \leqslant x \leqslant M$, $M>0$ and for all r in $\mathcal{E}$ and $f(x,r)=M/\delta$ for $x>M$. (We bound $f(x,r)$ from above so that it satisfies the assumptions we placed earlier on our production function, $f(x,r)$).

Let $\bar{y}\in(0,M)$, and let $\{x_t\}_{t=0}^{\infty}$ be an optimal input process from y. Suppose $x_1 < 1/\delta x_0$ and $x_0 < \bar{y}$.

Let $0 \leqslant \varepsilon \leqslant \bar{y} - x_0$. Construct a process $\{c_t', x_t'\}_0^{\infty}$ as follows:

$$x_0' = x_0 + \varepsilon,\ c_0' = c_0 - \varepsilon$$

$$c_1' = \frac{1}{\delta}(x_0+\varepsilon) - x_1,\ x_1' = x_1$$

$$\{c_t', x_t'\} = \{c_t, x_t\} \text{ for all } t \geqslant 2$$

Then clearly $\{c_t', x_t'\}$ is feasible from $\bar{y}$. Furthermore,

$$\sum_{t=0}^{\infty} \delta^t u(c_t) - \sum_{t=0}^{\infty} \delta^t u(c_t') = u(c_0) + \delta u(c_1) - u(c_0') - \delta u(c_1')$$

$$= \frac{1}{\delta}(c_0) + \delta\left(\frac{1}{\delta}\right)\left(\frac{1}{\delta}x_0 - x_1\right) - \frac{1}{\delta}(c_0-\varepsilon) - \delta\left(\frac{1}{\delta}\right)\left(\frac{1}{\delta}(x_0+\varepsilon) - x_1\right) = 0$$

Hence, the process $\{x_t'\}_0^{\infty}$ is optimal. In particular if x_0 is optimal from y, and $x_0 \leqq x_0' \leqq y$ then x_0' is also optimal from y. Thus $h(y)=y$ for y in $[0,M]$ is an optimal policy function.

Under the assumption that $x_0 < \bar{y}$ we obtain the following as an optimal policy function, on $[0,M)$.

$$h(y) = y \text{ for all } y \neq \bar{y}$$

$$h(\bar{y}) = x_0 < \bar{y}.$$

Clearly this optimal investment policy function is *not* monotone non-decreasing.

If $x_0 = \bar{y}$ and $x_1 < 1/\delta x_0$ we can use the above method and define

$$\varepsilon = \frac{x_0 - \delta x_1}{2},\ x_0' = \bar{y} - \varepsilon < \bar{y},\ h(y) = y \text{ for all } y \neq \bar{y} \text{ and } h(\bar{y}) = x_0' < \bar{y}$$

and obtain the result that $h(y)$ is *not* monotone non-decreasing.

Finally, we note that $x_1 = 1/\delta x_0$ *and* $x_0 = \bar{y}$ cannot hold for any $\bar{y} \in (\bar{M}, M)$ where $\bar{M} = \delta M$ (see Fig. 19.2)

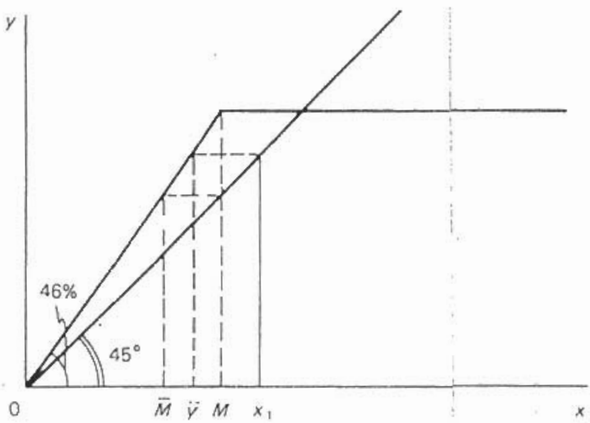

Figure 19.2

For in the case notice that x_1 lies in the horizontal flat portion of the production function. Thus we may put $x_1' = x_1 - \varepsilon$ with $\varepsilon > 0$ small enough so that x_1' also lies in the horizontal flat portion. Then output in the next period, period two, is unchanged; hence, defining $c_1' = \varepsilon$ and $\{x_t', c_t'\} = \{x_t, c_t\}$ for $t > 1$ we obtain a process with higher expected discounted total utility.

Example 2: Now we present an example where the production function is not uniformly bounded away from zero for each positive input level, and as a result, Theorem 5 (Stochastic Ramsey–Euler Condition) ceases to remain valid. Suppose $u(c) = -1/c$, $f(x,r) = rx^{1/2}$ and r is uniformly distributed on [0,1]. Then, since $c(f(h(y),r)) \leq f(h(y),r)$, if the stochastic Ramsey–Euler condition holds,

$$u'(c(y)) \geqq \delta \int u'(f(h(y),r))f'(h(y),r)d\gamma = \frac{\delta}{2}[h(y)]^{3/2} \int_0^1 \frac{1}{r} dr = \infty$$

Example 3: Here we show that without (T.12), Theorem 18 may fail. Let $u(c)=c^{1/2}$, $f(x,r)=f(x)=x^4$ for $0 \leqq x \leqq 1$, $y=\frac{1}{2}$, $\delta=\frac{1}{2}$. If $(\underset{\sim}{x},\underset{\sim}{c},\underset{\sim}{y})$ is any process from y, since $y<1$ and $f(x,r) \leqq x$ for $0 \leqq x \leqq 1$, then $0 \leqq x_t \leqq 1$ for each t, hence the definition of $f(x,r)$ for $x>1$ is unimportant in the discussion below. Notice that (T.12) does not hold. Define the process $(\underset{\sim}{x}^*,\underset{\sim}{c}^*,\underset{\sim}{y}^*)$ from initial stock y, by $y_0^*=c_0^*=y$, $x_0^*=0$ and for all $t \geqq 1$, $x_t^*=c_t^*=y_t^*=0$. We will show that $(\underset{\sim}{x}^*,\underset{\sim}{c}^*,\underset{\sim}{y}^*)$ is optimal from initial stock y, hence Theorem 18 does not hold.

The discounted total utility of $(\underset{\sim}{x}^*,\underset{\sim}{c}^*,\underset{\sim}{y}^*)$ is $(\frac{1}{2})^{1/2}$. Let $(\underset{\sim}{x},\underset{\sim}{c},\underset{\sim}{y})$ be any process from initial stock y with some fixed input level $x_0>0$. We shall show that the discounted total utility of $(\underset{\sim}{x},\underset{\sim}{c},\underset{\sim}{y})$ is less than $(\frac{1}{2})^{1/2}$. Define the total accumulation process $\{\bar{x}_t\}$ from input x_0 by $\bar{x}_0=x_0$ and $\bar{x}_t=f(\bar{x}_{t-1})$ for all $t \geqq 1$. Notice that $\bar{x}_t=x_0^{4^t} \leqq x_0^{4t}$ for $t \geqq 1$. Also $c_t \leqq \bar{x}_t$ for $t \geqq 1$, hence, the process $(\underset{\sim}{x},\underset{\sim}{c},\underset{\sim}{y})$ has discounted total utility less than

$$(\tfrac{1}{2}-x_0)^{\frac{1}{2}}+\sum_{t=1}^{\infty}(\tfrac{1}{2})^t(x_0^{4t})^{\frac{1}{2}}=(\tfrac{1}{2}-x_0)^{\frac{1}{2}}+\frac{\frac{1}{2}x_0^2}{1-\frac{1}{2}x_0^2} \tag{24}$$

The difference between the discounted total utility of $(\underset{\sim}{x}^*,\underset{\sim}{c}^*,\underset{\sim}{y}^*)$ and $(\underset{\sim}{x},\underset{\sim}{c},\underset{\sim}{y})$ exceeds

$$D(x_0) \equiv (\tfrac{1}{2})^{\frac{1}{2}}+q_1(x_0)+q_2(x_0) \tag{25}$$

where $q_1(x_0)=-(\frac{1}{2}-x_0)^{\frac{1}{2}}$ and $q_2(x_0)=-\dfrac{\frac{1}{2}x_0^2}{1-\frac{1}{2}x_0^2}$

To show that $(\underset{\sim}{x}^*,\underset{\sim}{c}^*,\underset{\sim}{y}^*)$ is optimal, it suffices to show that $D(x_0)>0$ for $x_0>0$. Notice that $D(0)=0$, so we only have to show that $D'(x)>0$ for $0<x \leqq \frac{1}{2}$. However, one can check that $q_1(x)$ is convex so $q_1'(x) \geqq q_1'(0)=(\frac{1}{2})^{\frac{1}{2}}>1/1.5$ and also that $q_2(x)$ is concave so $q_2'(x) \geqq q_2'(\frac{1}{2})>-1/1.5$ for $0<x \leqq \frac{1}{2}$. Addition then yields $D'(x)>0$ for $0<x \leqq \frac{1}{2}$. So $(\underset{\sim}{x}^*,\underset{\sim}{c}^*,\underset{\sim}{y}^*)$ is optimal and hence Theorem 18 fails to hold.

8 PROOFS

Proof of Theorem 1: One may readily adapt the basic existence theorem of Maitra (1968) to obtain Theorem 1. We now indicate how to obtain Theorem 1 using (U.1′) in place of (U.1) [see Remark 2 following the statement of Theorem 1]. Since S is compact, (U.1′) implies u is bounded above on S, hence, using Lemma 2, one may check that all the conditions of Schäl (1975, theorem 16.1) hold, so we conclude there exists an optimal stationary policy $\underset{\sim}{\pi}^*=(\hat{h}^{(\infty)})$ where $\hat{h}$: $S \rightarrow A$ is a Borel measurable function. The functional equation (eqs (8) and (9)) follow from the same theorem. From Schäl (1975

Corollary 6.3) we obtain that the value function, V, is upper semicontinuous (u.s.c.). Since, clearly, V is monotone non-decreasing, this implies that V is right continuous. To show that V is left continuous, fix any $y>0$, and suppose x^* is an optimal input from y, and $c^*=y-x^*$. Assume $c^*>0$, for otherwise $V(y)=-\infty$ and left continuity of V at y is trivial. Next, let $y_n \uparrow y$, so that for large enough n, $y_n - x^* \geqq 0$. Then, from the functional equation,

$$V(y)=u(c^*)+\delta \int V(f(x^*,r))\gamma(dr)$$

$$V(y_n) \geqq u(y_n - x^*) + \delta \int V(f(x^*,r))\gamma(dr)$$

hence, $V(y_n)-V(y) \geqq u(y_n - x^*) - u(c^*)$, and taking limits gives

$$\lim_{n\to\infty} V(y_n) \geqq V(y),$$

so combining with the fact that V is u.s.c. yields the left continuity of V.

QED

Proof of Theorem 2: We will now prove a stronger version of Theorem 2. Theorem 2(S): Suppose $y>y'$, and $\{x_t\}_0^\infty$, $\{x_t'\}_0^\infty$ are optimal input processes from y,y', respectively. Then $x_0 \geqq x_0'$.

Proof: Theorem 2 follows from Theorem 2(S) by defining $x_0=h(y)$ and $x_0'=h(y')$.

Suppose, on the contrary, that $x_0<x_0'$. Define new input processes from y,y' as follows. Let $\bar{x}_t=x_t'$ for all $t\geqq 0$. Then $\bar{x}_0=x_0'\leqq y'<y$, and for $t\geqq 1$, $\bar{x}_t=x_t'\leqq f(x_{t-1}',r_t)=f(\bar{x}_{t-1},r_t)$, hence $\{\bar{x}_t\}$ is feasible from y. Next, define $\bar{x}_t'=x_t$ for $t\geqq 0$. Then $\bar{x}_0'=x_0<x_0'\leqq y'$ and for $t\geqq 1$, $\bar{x}_t'=x_t\leqq f(x_{t-1},r_t)=f(\bar{x}_{t-1}',r_t)$, hence $\{\bar{x}_t'\}$ is feasible from y'. Let $\{\bar{c}_t\}$, $\{\bar{c}_t'\}$ be the consumption processes corresponding to $\{\bar{x}_t\}$, $\{\bar{x}_t'\}$, respectively.

Using the functional equation (see eqs (10) and (11)) we obtain

$$u(c_0)+\delta\int V(f(x_0,r))\gamma(dr)=V(y)\geqq u(\bar{c}_0)+\delta\int V(f(\bar{x}_0,r))\gamma(dr)$$

$$u(c_0')+\delta\int V(f(x_0',r))\gamma(dr)=V(y')\geqq u(\bar{c}_0')+\delta\int V(f(\bar{x}_0',r))\gamma(dr) \tag{26}$$

Adding these two inequalities and noting that $x_0=\bar{x}_0'$, $x_0'=\bar{x}_0$ we obtain

$$u(c_0)+u(c_0')\geqq u(\bar{c}_0)+u(\bar{c}_0') \tag{27}$$

Now, $\bar{c}_0=y-x_0'>y'-x_0'=c_0'$ and $\bar{c}_0=y-x_0'<y-x_0=c_0$, so there is a $0<\theta<1$ such that $\bar{c}_0=\theta c_0+(1-\theta)c_0'$. Then $\bar{c}_0'=y'-x_0=(y-x_0)+(y'-x_0')-(y-x_0')=c_0+c_0'-\bar{c}_0=(1-\theta)c_0+\theta c_0'$. This gives, using the strict concavity of u,

$$u(\bar{c}_0) > \theta u(c_0) + (1-\theta)u(c_0') \tag{28}$$

and

$$u(\bar{c}_0') > (1-\theta)u(c_0) + \theta u(c_0') \tag{29}$$

so by addition

$$u(\bar{c}_0) + u(\bar{c}_0') > u(c_0) + u(c_0') \tag{30}$$

(30) contradicts (27), and proves that $x_0 \geqq x_0'$ for $y > y'$.

QED

Remark: The proof of Theorem 2 above is a modification of the proof of Dechert and Nishimura (1983, theorem 1).

Proof of Theorem 3

(a) $\varphi(y)$ is the set

$$\{x\text{: } x \text{ solves } \max_{0 \leqq x \leqq y} \; u(y-x) + \delta \int V(f(x,r))\gamma(dr)\}$$

Since both u, and V are continuous, (a) follows from the Maximum Theorem (see, e.g., Berge (1963), p.116). (b) $\varphi(y)$ is a subset of S, hence is bounded. For fixed y in S, since φ is upper semicontinuous, $\varphi(y)$ is closed. $\underline{h}(y)$ is therefore the minimum of the compact set $\varphi(y)$, and hence is well defined.

Next, we show that $\underline{h}(y)$ is monotone non-decreasing. Notice that if u were strictly concave then the result would follow from Theorem 2(S).

If u is concave (not necessarily strictly) we modify the proof in Theorem 2(S) as follows. Suppose $y > y'$ and let $\{x_t\}$, $\{x_t'\}$ be the optimal input processes from y, y', respectively, using $\underline{h}$. Without strict concavity of u, eqs (28), (29) and hence (30) hold only with weak inequalities, in the proof of Theorem 2. However, we may still derive a contradiction to (27) by noting that (27) in this case holds with strict inequality, since $\bar{x}_0' = x_0 < x_0' = \underline{h}(y') = \min \varphi(y')$. Hence, $\bar{x}_0'$ is not optimal from y', so (26), and therefore (27), hold with strict inequality. Hence, $\underline{h}$ is monotone non-decreasing.

Next, we show that $\underline{h}(y)$ is left continuous. Let $y_n \uparrow y_0$. Since $\underline{h}(y_n)$ is monotone non-decreasing and bounded above we obtain that

$$x'' \equiv \lim_{n \to \infty} \underline{h}(y_n)$$

exists. But for all n, $\underline{h}(y_n) \leqq \underline{h}(y_0)$, so

$$x'' \leqq \underline{h}(y_0) \tag{31}$$

Next, let $x_n = \underline{h}(y_n) \in \varphi(y_n)$, and notice that $x_n \to x''$, $y_n \to y_0$ and $x_n \in \varphi(y_n)$. Since φ is upper semicontinuous, $x'' \in \varphi(y_0)$. Hence,

$$x'' \geqq \min \varphi(y_0) = \underline{h}(y_0) \tag{32}$$

From (31) and (32),

$$\lim_{y_n \uparrow y_0} \underline{h}(y_n) = \underline{h}(y_0);$$

so $\underline{h}$ is left continuous.

(c) Proof is similar to that of (b) with obvious changes.

QED

Proof of Corollary 3: Choose $h(y) = \bar{h}(y)$ (or $\underline{h}(y)$ if left continuity is required), and apply Theorem 3.

QED

Proof of Theorem 4: First we show that under (T.4), $V(y) > -\infty$ for $y > 0$. Fix any $y > 0$. Given the $k > 0$ in (T.4), choose any $0 < x_0 < \min\{y, k\}$. Then the following input and consumption process is feasible: For each t, and history $\eta_t, x_t(\eta_t) = x_0$, $c_0 = y - x_0 > 0$ and for $t \geqq 1$, $c_t = f(x_{t-1}, r_t) - x_t = f(x_0, r_t) - x_0 > 0$. Using (T.6) and compactness of $\mathcal{E}$, there exists a $c' > 0$ such that $c_t \geqq c'$ for all $t \geqq 1$. Then

$$V(y) \geqq u(y - x_0) + \sum_{t=0}^{\infty} \delta^t u(c') = u(c_0) + \frac{\delta}{1-\delta} u(c') > -\infty \tag{33}$$

If from any $y > 0$, $c(y) = 0$, then (U.1′) implies $V(y) = -\infty$, contradicting (33). Hence, $c(y) > 0$ for $y > 0$. Finally, if $h(y) = 0$, then by (T.5), next period stock, and hence consumption, is zero, which by (U.1′) implies $V(y) = -\infty$, again contradicting (33). Hence, $h(y) > 0$, from which the theorem follows.

QED

Proof of Theorem 5: First we prove the following lemma.

Lemma 2A: If $A = [a_1, a_2]$ with $0 < a_1 < a_2 < \infty$, then

$$\inf_{y \in A} c(y) > 0.$$

Proof: Suppose, on the contrary, that

$\inf_{y \in A} c(y) = 0$ and suppose $y^n \in A$ and $\lim_{n \to \infty} c(y^n) = 0$.

Since A is compact we may assume without loss of generality that $y^n \to y^* \in A$. Define $\underline{c}(y) = y - \underline{h}(y)$ and $\bar{c}(y) = y - \bar{h}(y)$. If y^n contains a subsequence (retain the same index n) converging to y^* from below such that $y^n < y^{n+1}$ for all n, then by Theorem 2(*S*), $h(y^{n-1}) \leqq \underline{h}(y^n) \leqq h(y^{n+1})$ so taking limits as $n \to \infty$,

$$\lim_{n\to\infty} \underline{h}(y^n) = \lim_{n\to\infty} h(y^n) \text{ or } \lim_{n\to\infty} \underline{c}(y^n) = \lim_{n\to\infty} c(y^n) = 0.$$

Then using the left continuity of $\underline{c}(y)$ yields $\underline{c}(y^*) = 0$, which contradicts Theorem 4. A similar contradiction may be obtained if there is a subsequence $y^n \downarrow y^*$. This concludes the proof of the lemma.

We now proceed to prove Theorem 5. Fix $y^* > 0$, and define $c^* = c(y^*)$, $x^* = h(y^*)$ and $\bar{x}(r) = h(f(x^*,r))$. Define

$$H(c) \equiv u(c) + \delta \int u[f(y^* - c,r) - \bar{x}(r)]\gamma(dr) \tag{34}$$

for all c such that the expression is well defined. We begin with:

Claim 1: The following maximization problem (P) is solved by c^*:

$$\text{'max } H(c) \text{ subject to } 0 \leqq c \leqq y^* \text{ and } f(y^* - c,r) \geqq \bar{x}(r)\text{'} \tag{P}$$

To verify claim 1, note that if c^* did not solve (P), there is some $\hat{c}$ such that $0 \leqq \hat{c} \leqq y^*$, $f(y^* - \hat{c},r) \geqq \bar{x}(r)$ and $H(\hat{c}) > H(c^*)$. Define a new process $(\hat{c},\hat{x})$ as follows: $\hat{c}_0 = \hat{c}$, $\hat{x}_0 = y^* - \hat{c}$, $\hat{c}_1 = f(\hat{x}_0,r) - \bar{x}_1(r)$, $\hat{x}_1 = \bar{x}_1(r)$, $\hat{c}_t = c_t$, $\hat{x}_t = x_t$ for all $t \geqq 2$, where (c,x) is an optimal process from y^*. The difference between the expected total discounted utility of (c,x) and $(\hat{c},\hat{x})$ is $H(c^*) - H(\hat{c}) < 0$, a contradiction to optimality of (c,x). This establishes claim 1.

Claim 2: There exists a $\xi > 0$ such that for all c in $U \equiv (c^* - \xi, c^* + \xi)$, the contraints '$0 \leqq c \leqq y^*$' and '$f(y^* - c,r) - \bar{x}(r) \geqq 0$' are satisfied.

To prove claim 2, note that from Theorem 4, $0 < c^* < y^*$, so let $\xi_1 > 0$ be such that $0 < c^* - \xi_1 < c^* + \xi_1 < y^*$. Then for c in $U_1 \equiv (c^* - \xi_1, c^* + \xi_1)$ the first constraint in claim 2 is satisfied.

Since $\mathcal{E}$ is compact, (T.6) implies that the following are well defined and positive for $x > 0$;

$$f_m(x) \equiv \min_{r\in\mathcal{E}} f(x,r) \text{ and } f_M(x) \equiv \max_{r\in\mathcal{E}} f(x,r).$$

Then $f(x^*,r) \in [f_m(x^*), f_M(x^*)]$, so Lemma 2A implies there is a $k > 0$ such that

$$f(x^*,r) - h(f(x^*,r)) \geqq k \quad \gamma - \text{a.e.} \tag{35}$$

f is continuous on the compact set $S \times \mathcal{E}$ and so is uniformly continuous

$$\sup_r |f(x^*+\varepsilon,r)-f(x^*,r)| < \frac{k}{2}$$

for all ε in $(-\xi,\xi)$. Define $U \equiv (c^*-\xi, c^*+\xi)$. Then for any c in U, putting $\varepsilon = c^*-c$, we obtain

$$\begin{aligned} f(y^*-c,r)-\bar{x}(r) &= f(x^*+\varepsilon,r)-\bar{x}(r) = f(x^*+\varepsilon,r)-h(f(x^*,r)) \\ &= [f(x^*,r)-h(f(x^*,r))]+[f(x^*+\varepsilon,r)-f(x^*,r)] > k-\frac{k}{2}=\frac{k}{2}; \end{aligned}$$

i.e. for c in U

$$f(y^*-c,r)-\bar{x}(r) > \frac{k}{2} \quad \gamma \text{ a·e} \tag{36}$$

In particular the second constraint is satisfied for each c in U. Since $U \subset U_1$, the first constraint is also satisfied for each c in U. This completes proof of claim 2.

For c in U and any ε such that $c+\varepsilon$ is also in U, $[H(c+\varepsilon)-H(c)]/\varepsilon = A(c,\varepsilon)+\delta\int D(c,\varepsilon,r)\gamma(dr)$ where $A(c,\varepsilon)=[u(c+\varepsilon)-u(c)]/\varepsilon$ and $D(c,\varepsilon,r) = [u(f(y^*-c-\varepsilon,r)-\bar{x}(r))-u(f(y^*-c,r)-\bar{x}(r))]/\varepsilon$.

Claim 3: There is an $M<\infty$ such that for each c in U and for any ε such that $c+\varepsilon$ is in U, $|D(c,\varepsilon,r)| \leqq M$ $\gamma-$a.e.

To prove claim 3, suppose c is in U, and ε is such that $c+\varepsilon$ is in U. Then (36) implies for some $k>0$, $f(y^*-c-\varepsilon,r)-\bar{x}(r) \geqq k/2$ and $f(y^*-c,r)-\bar{x}(r) \geqq k/2$ γ a.e. Since $y^*-c-\varepsilon \leqq y^*-c^*+\xi$ we obtain $f(y^*-c-\varepsilon,r)-\bar{x}(r)$ and $f(y^*-c,r)-\bar{x}(r)$ both lie in the interval $[k/2, f_M(y^*-c^*+\xi)]$ where

$$f_M(x) \equiv \max_{r\in\mathcal{E}} f(x,r).$$

Hence, (U.4) implies there is an $M_1>0$ such that for each c and $c+\varepsilon$ in U,

$$\left|\frac{u(f(y^*-c-\varepsilon,r)-\bar{x}(r))-u(f(y^*-c,r)-\bar{x}(r))}{f(y^*-c-\varepsilon,r)-f(y^*-c,r)}\right| \leqq M_1 \quad \gamma-\text{a.e.} \tag{37}$$

Also, since $y^*-c-\varepsilon$ and y^*-c both lie in $[x^*-\xi, x^*+\xi]$, (T.7) implies there is an $M_2<\infty$ such that for all c and $c+\varepsilon$ in U,

$$\left|\frac{f(y^*-c-\varepsilon,r)-f(y^*-c,r)}{\varepsilon}\right| \leqq M_2 \quad \gamma-\text{a.e.} \tag{38}$$

But (37) and (38) imply $|D(c,\varepsilon,r)| \leq M_1M_2$ [illegible]

Now

$$\lim_{\varepsilon\to 0} A(c,\varepsilon)=u'(c).$$

Also

$$\lim_{\varepsilon\to 0} D(c,\varepsilon,r)=-u'(f(y^*-c,r)-\bar{x}(r))f'(y^*-c,r),$$

so using claim 3 above and the Dominated Convergence Theorem yields

$$\lim_{\varepsilon\to 0}\frac{H(c+\varepsilon)-H(c)}{\varepsilon}=u'(c)-\delta\int u'(f(y^*-c,r)-x(\bar{r}))f'(y^*-c,r)\gamma(dr) \quad (39)$$

Hence, $H(c)$ is differentiable at each c in U, and $H'(c)$ equals the expression on the right of (39).

Finally, since c^* solves the problem 'maximize $H(c)$ subject to $c^*-\xi<c<c^*+\xi$', and H is differentiable on the open constraint set, we obtain by the classical first-order conditions of calculus that $H'(c^*)=0$ and the stochastic Ramsey–Euler condition follows immediately.

QED

Proof of Corollary 5: $h(y)$ from Theorem 2 is monotone non-decreasing. Suppose there exists $\underline{y}$, $\bar{y}$ with $\bar{y}>\underline{y}>0$ such that $h(\underline{y})=h(\bar{y})\equiv h^*$ (say). Then for all $y\in[\underline{y},\bar{y}]$, $H(y)=h^*$. From the stochastic Ramsey–Euler conditions, this implies

$$u'(c(y))=\delta\int u'(c(f(h(y),r)))f'(h(y),r)\gamma(dr)$$

$$=\delta\int u'(c(f(h^*,r)))f'(h^*,r)\gamma(dr)=m^*, \text{ say, for all } y\in[\underline{y},\bar{y}].$$

Hence, since u is strictly concave, $c(y)=c^*$ (say) for all y in $[\underline{y},\bar{y}]$, so $y=h(y)+c(y)=h^*+c^*=y^*$ (say) for all y in $[\bar{y},\underline{y}]$ which is a contradiction, proving that $h(y)$ must be strictly increasing.

QED

We shall prove Theorem 6 after we have proved all the results of Section 5.

Proof of Lemma 3: Let $E=\{y\in S|\bar{h}(y)-\underline{h}(y)>0\}$. Given any integer n, let $E_n=\{y\in S|\bar{h}(y)-\underline{h}(y)>1/n\}$. Then $E_n\uparrow E$ as $n\uparrow\infty$. Let $C=\{y\in S|\underline{h}(y)$ is continuous at $y\}$. Since $\underline{h}(y)$ is monotone non-decreasing, $C'=S-C$ is at most countable (see, e.g., K. L. Chung (1974), p.4). Fix an n and let $y_0\in C\cap E_n$, and $y_k\downarrow y_0$. Then

$$\lim_{k\to\infty} \underline{h}(y_k) = \underline{h}(y_0)$$

since $y_0 \in C$; so there is a $k^* = k^*(n)$ such that for all $k > k^*$,

$$\underline{h}(y_k) \leqq \underline{h}(y_0) + \frac{1}{2n} < \underline{h}(y_0) + \frac{1}{n} \leqq \bar{h}(y_0). \tag{40}$$

The last inequality is from the hypothesis that $y_0 \in E_n$. Pick any $k > k^*$ and let $y' = y_k$, $x' = \underline{h}(y_k)$ and $x_0 = \bar{h}(y_0)$. Then (40) tells us that even though $y' > y$ we have $x' < x$. This contradicts Theorem 2(*S*); so for all n, $C \cap E_n$ is empty. Thus $E_n \subset S - C = C'$, so E_n is at most countable.

Finally, since

$E = \bigcup_{n=1}^{\infty} E_n$, E is a countable union of countable sets and hence is countable.

Proof of Lemma 4: Fix $y > 0$ and let $\underline{y}^n \uparrow y$. Let $\{\underline{x}_t^n\}_0^\infty$ be the optimal input process from $\underline{y}^n$ obtained using the optimal investment policy function $\underline{h}$. We proceed to show

$$u'(y - x_0) \leqq \liminf_{n\to\infty} \frac{V(y) - V(y^n)}{y - y^n} \leqq \limsup_{n\to\infty} \frac{V(y) - V(y^n)}{y - y^n} \leqq u'(y - x_0)$$

which implies $V^-(y) = u'(y - \underline{x}_0)$. By the left continuity of $\underline{h}(y)$, $\underline{x}_0^n = \underline{h}(\underline{y}^n) \uparrow \underline{h}(y) = \underline{x}_0$ where $\{\underline{x}_t\}_0^\infty$ is the optimal input process from y (using $\underline{h}$ again). By Theorem 4, $\underline{x}_0 < y$. Thus $\underline{y}^n \uparrow y$ implies that there is an n^* such that $\underline{x}_0 \leqq \underline{y}^n$ for all $n > n^*$. So by the functional equation

$$V(y) = u(y - \underline{x}_0) + \delta \int V(f(\underline{x}_0, r))\gamma(dr).$$

$$V(\underline{y}^n) \geqq u(\underline{y}^n - \underline{x}_0) + \delta \int V(f(\underline{x}_0, r))\gamma(dr) \text{ for } n > n^*.$$

This leads to

$$V(y) - V(\underline{y}^n) \leqq u(y - \underline{x}_0) - u(\underline{y}^n - \underline{x}_0)$$

$$\leqq u'(\underline{y}^n - \underline{x}_0)[y - \underline{y}^n] \text{ from concavity of } u.$$

So

$$\limsup_{n\to\infty} \frac{V(y) - V(y^n)}{y - y^n} \leqq u'(y - x_0) \tag{41}$$

Since $\underline{h}$ is monotone non-decreasing,

$$x_0^n = h(y^n) \leq h(y) \leq y.$$

Hence, by the functional equation

$$V(y) \geqq u(y - \underline{x}_0^n) + \delta \int V(f(\underline{x}_0^n, r))\gamma(dr).$$

$$V(\underline{y}^n) = u(\underline{y}^n - \underline{x}_0^n) + \delta \int V(f(\underline{x}_0^n, r))\gamma(dr).$$

We then obtain $V(y) - V(\underline{y}^n) \geqq u(y - \underline{x}_0^n) - u(\underline{y}^n - \underline{x}_0^n) \geqq u'(y - \underline{x}_0^n)[y - \underline{y}^n]$ from concavity of u, so

$$\liminf_{n\to\infty} \frac{V(y) - V(\underline{y}^n)}{y - \underline{y}^n} \geqq u'(y - x_0). \tag{42}$$

Combining (41), (42) yields $V^-(y) = u'(y - \underline{x}_0)$. Similarly, by letting $\bar{y}^n \downarrow y$ and using the optimal investment policy function $\bar{h}$ one obtains $V^+(y) = u'(y - \underline{x}_0)$ where $\bar{x}_0 = \bar{h}(y)$. Hence, the right and left derivatives exist. Further using $\bar{h} \geqq \underline{h}$ and concavity of u,

$$V^-(y) = u'(y - \underline{h}(y)) \leqq u'(y - \bar{h}(y)) = V^+(y) \tag{43}$$

To prove the last part of the theorem, we obtain from Lemma 3 that $\underline{h}(y) = \bar{h}(y)$ except for countably many values of y. Hence, except for those countable values of y, $\underline{x}_0 = \underline{h}(y) = \bar{h}(y) = \bar{x}_0$, so $V^-(y) = u'(y - \underline{x}_0) = u'(y - \bar{x}_0) = V^+(y)$, and $V'(y)$ exists outside this countable set.

QED

Proof of Theorem 7: Suppose from $y > 0$, $\{x_0, x_1, x_2, \ldots\}$ and $\{x_0, x_1', x_2', \ldots\}$ are optimal input processes. By induction, it suffices to prove that $x_1 = x_1'$ a.s. Suppose, *ad absurdum*, that $\bar{h}(f(x_0, r)) > \underline{h}(f(x_0, r))$ with strictly positive γ probability. Define $\bar{c}(y) = y - \bar{h}(y)$ and $\underline{c}(y) = y - \underline{h}(y)$. Then, under the hypotheses of the stochastic Ramsey–Euler condition, since $\bar{c}(f(x_0, r))$, $\underline{c}(f(x_0, r))$ are *both* optimal consumptions from $f(x_0, r)$, we obtain

$$u'(y - x_0) = \delta \int u'(\bar{c})f(x_0, r)))f'(x_0, r)\gamma(dr)$$

and

$$u'(y - x_0) = \delta \int u'(\underline{c}(f(x_0, r)))f'(x_0, r)\gamma(dr)$$

so

$$\int [u'(\bar{c}(f(x_0, r))) - u'(\underline{c}(f(x_0, r)))]f'(x_0, r)\gamma(dr) = 0. \tag{44}$$

But $\bar{c}(f(x_0, r)) \leqq \underline{c}(f(x_0, r))$ for each r, with strict inequality holding with γ positive probability. Since u is strictly concave (U.2), and f is assumed increasing (T.3), the left-hand side of (44) is strictly positive. This is a contradiction, and proves that $\underline{h}(f(x_0,r))=\bar{h}(f(x_0,r)$ so $x_1=x_1'$ a.s.

We now prove the second part of the theorem. If the optimal path is uniquely determined then $\bar{x}_0=\bar{h}(y)=\underline{h}(y)=\underline{x}_0$, hence from equation (43) above we see that $V^-(y)=V^+(y)$ so V is differentiable.

If V is differentiable, since u is strictly concave we obtain from eq. (43) that $\bar{x}_0=\bar{h}(y)=\underline{h}(y)=\underline{x}_0$. Then the first part of this theorem implies that the entire input process is uniquely determined.

QED

Proof of Theorem 8: From eq. (43), for $y>0$, since $y-\bar{h}(y)\leqq c(y)\leqq y-\underline{h}(y)$, we obtain $V^-(y)=u'(y-\underline{h}(y))\leqq u'(c(y))\leqq u'(y-\bar{h}(y))=V^+(y)$. If V is differentiable at y, then $V'(y)=u'(c(y))$, which is the first equality in eq. (14).

Since V is differentiable at $y>0$, Theorem 7 implies that there is a unique optimal process from y; so for γ a.e. r, there is a unique optimal process from $y_1=f(h(y),r)$; hence, using Theorem 7 again, V is differentiable at $y_1=f(h(y),r)$. The second equality in (7) then follows immediately from the stochastic Ramsey–Euler condition, and the first equality (replacing y with $y_1=f(h(y),r)$).

QED

Proof of Theorem 6: Concavity of V implies that for all $y>0$, $V^-(y)\geqq V^+(y)$. Combining this with equation (43) proves the first assertion. Next, suppose that V is concave. Then V is differentiable, so from Theorem 8, $V'(y)=u'(c(y))$. Monotonicity of $c(y)$ then follows from concavity of V and u. Finally, if $c(y)$ is monotone non-decreasing, to prove that V is concave, it suffices to show that $u'(c(y))=V'(y)$ for all $y>0$. Let C be the set of points $y>0$ where the equality does not hold. From Lemma 4 and Theorem 8, C is the set of points where V is not differentiable, and C is at most countable. To complete the proof we will show that C is empty. Suppose, instead, that $y\in C$. Since C is countable, we may choose sequences $\{\underline{y}_n\}$, $\{\bar{y}_n\}$ such that $\underline{y}_n\uparrow y$, $\bar{y}_n\downarrow y$ with $\underline{y}_n,\bar{y}_n$ outside C for all n. Using eq. (43) and the right continuity of $\bar{h}$, gives

$$V^+(y)=u'(y-\bar{h}(y))=\lim_{n\to\infty}u'(\bar{y}_n-\bar{h}(\bar{y}_n))=\lim_{n\to\infty}V'(\bar{y}_n).$$

Similarly, we may show

$$V^-(y)=\lim_{n\to\infty}V'(\underline{y}_n).$$

Monotonicity of $c(y)$ implies $V'(\bar{y}_n)=u'(c(\bar{y}_n))\leqq u'(c(\underline{y}_n))=V'(\underline{y}_n)$, so taking limits $V^+(y)\leqq V^-(y)$, which from eq. (43) means V is differentiable at $y\in C$, contradicting the definition of C. Hence, C is empty.

Proof of eq. (21):

$$H_m(x)=h(f_m(x)),\ H_M(x)=h(f_M(x)) \tag{21}$$

We prove only the first equation, the second following similarly. By definition, $f_m(x)\leqq f(x,r)$ for all r, hence from monotonicity of h, $h(f_m(x))\leqq h(f(x,r))$ for all r, so

$$h(f_m(x))\leqq \min_{r\in\mathcal{E}} h(f(x,r))\equiv h_m(x) \tag{45}$$

Next, since $\mathcal{E}$ is finite,

$$\min_{r\in\mathcal{E}} f(x,r)$$

is attained at some r'; i.e. $f_m(x)=f(x,r')$. Hence,

$$h(f_m(x))=h(f(x,r'))\geqq \min_{r\in\mathcal{E}} h(f(x,r))\equiv h_m(x) \tag{46}$$

From (45) and (46) we obtain the first equation in (21).

QED

Proof of Theorem 9: One checks that if $H(\cdot,r)$ is monotone non-decreasing (but not necessarily continuous) γ a.e., one may still apply Dubins and Freedman (1966, corollary 5.5) [see, e.g., Bhattacharya, 1985].

Proof of Lemma 5

(a) Suppose for some fixed r, x^* is a fixed point of $H(\cdot,r)$ but is not a fixed point of one of $\bar{H}(\cdot,r)$, $\underline{H}(\cdot,r)$. Then $H(x^*,r)=x^*$ and $\bar{H}(x^*,r)>\underline{H}(x^*,r)$. Let $y=f(x^*,r)$. Define two processes $(\underline{x},\underline{c},\underline{y})$ and $(\underline{x}',\underline{c}'\underline{y}')$ from y as follows: the process $(\underline{x},\underline{c},\underline{y})$ is obtained by using h in the initial period, and $\underline{h}$ in each subsequent period. The process $(\underline{x}',\underline{c}'\underline{y}')$ is obtained using the policy h in the initial period, and the policy function $\bar{h}$ in each subsequent period.

Then $(\underline{x},\underline{c},\underline{y})$ and $(\underline{x}',\underline{c}'\underline{y}')$ are both optimal processes from the initial stock $y>0$, and $x_0=x_0'=h(y)=h(f(x^*,r))\equiv H(x^*,r)=x^*$. Let $x_1(r)$, $x_1'(r)$ denote the inputs in period one corresponding to the processes $(\underline{x},\underline{c},\underline{y})$, $(\underline{x}',\underline{c}'\underline{y}')$, respectively, when the shock occurring in period one is r. Then

$$x_1(r)=\underline{h}(f(x_0,r))=\underline{h}(f(x^*,r))\equiv \underline{H}(x^*,r)<\bar{H}(x^*,r)\equiv \bar{h}(f(x^*,r))$$

$$=\bar{h}(f(x_0',r))=x_1'(r).$$

Under (E.1), the shock r occurs with positive probability, so we obtain a contradiction to the uniqueness result of Theorem 7.

(b) Suppose that $x_0 = H_M(x_0)$, and that the maximum defining $f_M(x_0)$ is attained at r_M (see eq. (21)), i.e. $f(x_0,r_M) = f_M(x_0)$. Then $x_0 = H_M(x_0) = h(f(x_0,r_M))$; hence, by (a) above, $\bar{H}(x_0,r_M) = \underline{H}(x_0,r_M) = x_0$ which gives, using eq. (21), $\bar{H}_M(x_0) = \underline{H}_M(x_0) = x_0$. Therefore, H_M, $\bar{H}_M$, $\underline{H}_M$ all have the same fixed points. Proof of (c) is similar to (b).

QED

Proof of Lemma 6: We require the following:

Claim 1: Suppose $G: S \to S$ is monotone non-decreasing and there exists $x_1 < x_2$ in S such that $G(x_1) > x_1$ and $G(x_2) < x_2$. Then there exists an x_3 in (x_1,x_2) with $G(x_3) = x_3$.

The claim can be proved using a simple sequential argument.

Claim 2: Fix an r in $\mathcal{E}$. Then $H(\cdot,r)$ cannot have a sequence of positive fixed points $\{x_n\}$ such that $x_n \to 0$ as $n \to \infty$.

Proof: Fix an r *in* $\mathcal{E}$. Suppose, contrary to the claim, that $H(x_n,r) = x_n$ for each n, and $x_n \to 0$ as $n \to \infty$. From the stochastic Ramsey–Euler condition,

$$u'(c(f(x_n,r)) = \delta \int u'(c(f(H(x_n,r),\sigma)))f'(H(x_n,r),\sigma)\gamma(d\sigma)$$

$$= \int u'(c(f(x_n,\sigma)))f'(x_n,\sigma)\gamma(d\sigma)$$

$$\geqq \delta\gamma(\{r\})u'(c(f(x_n,r)))f'(x_n,r)$$

Hence, $\quad 1 \geqq \delta\gamma(\{r\})f'(x_n,r).$

Under (T.8), the Inada condition at the origin, the right-hand side of the above inequality tends to infinity as n tends to infinity, which is a contradiction, which completes proof of claim 2.

Now we prove Lemma 6. Since $\mathcal{E}$ is finite it suffices to show that for fixed r, there exists an $\varepsilon_r > 0$ with $H(x,r) > x$ for all x in $(0,\varepsilon_r)$. Fix an r in $\mathcal{E}$. Define $M' = \inf\{x > 0 | H(x,r) = x\}$, then by claim 2, $M' > 0$. Suppose, *ad absurdum*, that there is no $\varepsilon_r < M'$ such that $H(x,r) > x$ for each x in $(0,\varepsilon_r)$. Pick any M in $(0,M')$ such that $H(M,r) < M$. If there is a $z < M$ with $H(z,r) > Z$, then claim 1 implies there is a $z' > 0$ with $z < z' < M < M'$ such that $H(z',r) = z'$. This contradicts the definition of M'.

Therefore,

$$H(x,r) < x \text{ for each } x \text{ in } (0,M) \tag{47}$$

Next, we show there is a K in $(0,M)$ such that

$$c(f(h(y),r)) > c(y) \text{ for each } y \text{ in } (0,K) \tag{48}$$

To prove (48) pick any y in $(0,M)$. Then from the stochastic Ramsey–Euler condition

$$u'(c(y)) = \delta \int f'(h(y),\sigma)u'(c(f(h(y),\sigma)))\gamma(d\sigma)$$

$$\geqq \delta\gamma(\{r\})f'(h(y),r)u'(c(f(h(y),r)))$$

Hence,

$$\frac{1}{\delta\gamma(\{r\})f'(h(y),r)} \geqq \frac{u'(c(f(h(y),r)))}{u'(c(y))}.$$

Since $f'(h(y),r) \to \infty$ as $y \to 0$, the left-hand side of the above inequality tends to zero as $y \to 0$. Choose K in $(0,M)$ such that for all y in $(0,K)$,

$$\frac{u'(c(f(h(y),r)))}{u'(c(y))} < 1,$$

so by strict concavity of u, $c(f(h(y),r)) > c(y)$ and (48) follows.

Next, pick any $y_0 \in (0,K)$ and define $x_0 = h(y_0)$, $c_0 = y_0 - h(y_0)$ and for $n \geqq 1$, $x_n = H(x_{n-1},r)$, $y_n = f(x_{n-1},r)$ and $c_n = y_n - x_n$.

Claim 3: $H(x,r) < x$ for all x in $(0,M)$ [i.e. (47)], implies $x_n \to 0$, $y_n \to 0$ and $c_n \to 0$ as $n \to \infty$.

To prove claim 3 it suffices to show $x_n \to 0$ as $n \to \infty$. $x_0 \leqq y_0$ and $y_0 \in (0,M)$ implies $x_0 \in (0,M)$. If $x_{n-1} \in (0,M)$ then since $x_n = H(x_{n-1},r) < x_{n-1}$, we have $x_n \in (0,M)$. So by induction $x_n \in (0,M)$ for each n, hence $x_{n+1} = H(x_n,r) < x_n$ for each n, and $x_n \downarrow x^*$ (say). We will show $x^* = 0$. Using the monotonicity result of Theorem 2(S), since $x_{n+1} < x_n < x_{n-1}$, $\bar{H}(x_{n+1},r) \leqq H(x_n,r) \leqq \bar{H}(x_{n-1},r)$. Taking limits and using the right continuity of $\bar{H}$,

$$\bar{H}(x^*,r) = \lim_{n\to\infty} \bar{H}(x_n,r) = \lim_{n\to\infty} H(x_n,r) \equiv \lim_{n\to\infty} x_{n+1} = x^*. \tag{49}$$

Hence, x^* is a fixed point of $\bar{H}$. From Lemma 5(a), x^* is a fixed point of H. Clearly $x^* < M$. If $x^* > 0$, we obtain a contradiction to the definition of M; so $x^* = 0$, and this completes proof of claim 3.

However, by putting $y = y_n$ in (48) we obtain $c_{n+1} > c_n$ for each n, so $c_n > c_0$ for each n. By Theorem 4, $c_0 > 0$, hence c_n cannot converge to zero, contradicting claim 3. So for fixed r, there is an $\varepsilon_r > 0$ with $H(x,r) > x$ for all x in $(0,\varepsilon_r)$, and r is in $\mathcal{E}$, a finite set.

QED

Proof of Lemma 7

(a) We shall show only that y_m is well defined, the others following similarly. Suppose $x^k \downarrow y_m$ with $H_m(x^k) = x^k$ for all k. Then Lemma 5(c) and eq. (21) imply $x^k = \bar{H}_m(x^k) = x^k)$). Taking limits, and using the continuity of $f_m(x)$ and right continuity of $\bar{h}$, $y_m = \bar{h}(f_m(y_m)) = \bar{H}_m(y_m)$. Applying Lemma 5(c) again, we get $y_m = H_m(y_m)$, hence y_m is well defined.

(b) Follows from Lemma 6.

(c) If $H_m(x) < x$ for some x in $(0, y_m)$ then claim 1 in proof of Lemma 6, and Lemma 6 itself, would imply there is some $x' \in (x, y_m)$ such that $H_m(x') = x'$ and this contradicts definition of y_m. Hence, $H_m(x) > x$ for $x \in (0, y_m)$. Similarly, using (T.1) which implies $H_m(\beta) < \beta$, we can show that $H_m(x) < x$ for x in (x_m, ∞).

The proof of (d) is similar to that of (c). (e) follows immediately from (c), (d) and the fact that $H_m(x) \leqq H_M(x)$ for all x.

QED

Proof of Theorem 10: Let $S' = [c,d]$ and let $[a,b]$ be the unique minimal γ-invariant subset of S'. We proceed to show that for some integer m, γ^m splits, and in particular,

$$\gamma^m(\{r^m \varepsilon \mathcal{E}^m \mid H^m(x, r^m) \leqq \frac{a+b}{2} \text{ for all } x \text{ in } S'\}) > 0$$

and

$$\gamma^m(\{r^m \in \mathcal{E}^m \mid H^m(x, r^m) \geqq \frac{a+b}{2} \text{ for all } x \text{ in } S'\}) > 0 \tag{50}$$

Define
$a_n = \sup\{z \mid H^n(d, r^n) \geqq z, \text{ for all } r^n \in \mathcal{E}^n\}$ and $a^* = \inf_n a_n$. We will show $a^* = a$. First, since $[a,b]$ is γ-invariant, $H^n(d, r^n) \geqq H^n(b, r^n) \geqq a$ for all $r^n \in \mathcal{E}^n$. Hence, $a_n \geqq a$ for each n, and $a^* \geqq a$. Next, to show $a^* \leqq a$, we will prove that $[a^*, d]$ is γ-invariant, which implies, since $[a,b]$ is minimal, that $[a,b] \subset [a^*, d]$ so $a^* \leqq a$. To show $[a^*, d]$ is γ-invariant, it suffices to show

$$H(a^*, r) \geqq a^* \text{ for all } r \tag{51}$$

First we prove the following general lemma. Recall $\bar{H}(x,r) = \bar{h}(f(x,r))$ where $\bar{h}$ is defined in (12). From Theorem 3(c), $\bar{H}$ is right continuous.

Lemma 8

$$\bar{H}(a^*,r) \geqq a^* \text{ for each } r. \tag{52}$$

To prove Lemma 8, we will first prove the following claim:

Claim: Fix an n. Then for all r in some subset B_n of $\mathcal{E}$ with $\gamma(B_n)=1$

$$\bar{H}\left(a_n+\frac{1}{n},r\right) \geqq a_{n+1} \tag{53}$$

Proof: Suppose the claim is false. Then on some subset D of $\mathcal{E}$ with $\gamma(D)>0$,

$$a_{n+1} > \bar{H}\left(a_n+\frac{1}{n},r\right) \geqq H\left(a_n+\frac{1}{n},r\right).$$

By definition of a_n, for r^n in some subset A_n of $\mathcal{E}^n$ with $\gamma^n(A_n)>0$, $H^n(d,r^n) \leqq a_n+1/n$. Hence, for $r^{n+1}=(r^n,r)\in A_n\times D$,

$$H^{n+1}(d,r^{n+1}) \equiv H(H^n(d,r^n),r) \leqq H\left(a_n+\frac{1}{n},r\right) < a_{n+1},$$

which, since $\gamma^{n+1}(A_n\times D)=\gamma^n(A_n)\cdot\gamma(D)>0$, contradicts the definition of a_{n+1}. This completes the proof of the claim.

To prove Lemma 8, define

$$B=\bigcap_{n=1}^{\infty} B_n,$$

then $\gamma(B)=1$. Since $H(d,r)\leqq d\,\gamma$ a.e., by induction $H^{n+1}(d,r^{n+1})\leqq H^n(d,r^n)$ so $a_{n+1}\leqq a_n$ for each n, hence $a_n\downarrow a^*$. Taking limits in (53), and using the right continuity of $\bar{H}$ gives (52), for $r\in B$. This completes the proof of Lemma 8.

If H were continuous, then a simple modification of the proof of Lemma 8 results in (51). We now show

Lemma 9: The inequality (51) holds if we assume that $\mathcal{E}$ is finite and Lemma 5(a) holds.

Proof: Define $B_0=\{r\in\mathcal{E}\,|\,\underline{H}(a^*,r)<a^*\}$ and suppose *ad absurdum* that $\gamma(B_0)>0$. Since $\mathcal{E}$ is finite, for some r' *in* B_0,

$$M \equiv \max_{r\in B_0} f(a^*,r)=f(a^*,r'),$$

$$\underline{h}(M)=\underline{h}(f(a^*,r'))\equiv\underline{H}(a^*,r')<a^* \tag{54}$$

If $\bar{H}(a^*,r)=a^*$ for some r in B_0, we obtain an immediate contradiction to Lemma 5(a), so from (52),

$$\bar{H}(a^*,r)>a^* \text{ for each } r \text{ in } B_0 \tag{55}$$

and, in particular,

$$\bar{h}(M)=\bar{h}(f(a^*,r'))\equiv\bar{H}(a^*,r')>a^* \tag{56}$$

For r *not* in B_0, $\underline{h}(f(a^*,r))\equiv\underline{H}(a^*,r)\geqq a^*>\underline{h}(M)$; so from monotonicity of $\underline{h}$, $f(a^*,r)>M$, and hence by monotonicity of $\bar{h}$ and (56),

$$\bar{H}(a^*,r)\equiv\bar{h}(f(a^*,r))\geqq\bar{h}(M)>a^* \text{ for each } r \text{ not in } B_0 \tag{57}$$

From (55) and (57), $\bar{H}(a^*,r)>a^*$ for all r in $\mathcal{E}$. Since $\mathcal{E}$ is finite, there is some $\xi>0$ such that

$$\bar{H}(a^*,r)>a^*+\xi \text{ for all } r\in\mathcal{E} \tag{58}$$

and $a^*+\xi\leqq d$ (the latter inequality is possible since $a^*<d$; this follows from the fact that $H(d,r)\leqq d$ for all r, hence by (T.9), $H(d,r'')<d$ for some r'', hence $a^*\leqq a_1\leqq H(d,r'')<d$).

We will show that (58) leads to a contradiction. Define $\bar{a}=a^*+\xi$. Then for each r, $H(\bar{a},r)\equiv H(a^*+\xi,r)\geqq\bar{H}(a^*,r)>a^*+\xi=\bar{a}$ (where the first inequality follows from Theorem 2(*S*), and the second follows from (58)). Then $H(d,r)\geqq H(\bar{a},r)>\bar{a}$, so by induction, $H^n(d,r^n)\geqq\bar{a}$ for each r^n. Hence, $a_n\geqq\bar{a}$ for each n, so $a^*\geqq\bar{a}=a^*+\xi$, a contradiction. This shows that $H(a^*,r)\geqq \underline{H}(a^*,r)>a^*$, and concludes proof of Lemma 9.

Hence, we have shown that $a^*=a$. The proof of Theorem 10 then proceeds just as in Dubins and Freedman (1966, theorem, 5.15). We provide the rest of the proof here for the sake of completeness.

Since there are no γ-fixed points, $a<b$, so $a^*=a<a+b/2$. Hence, for some n_1, there is an $\bar{r}^{n_1}\in\mathcal{E}^{n_1}$ in the support of γ^{n_1} such that $H^{n_1}(d,\bar{r}^{n_1})<a+b/2$. Repeating a similar argument, there is an integer n_2 and an $\underline{r}^{n_2}\in\mathcal{E}^{n_2}$ in the support of γ^{n_2} such that $H^{n_2}(c,\underline{r}^{n_2})>a+b/2$.

Define $\bar{r}^{n_1+n_2}=(\underline{r}^{n_2},\bar{r}^{n_1})\in\mathcal{E}^{n_1+n_2}$, i.e. the first n_2 coordinates are $\underline{r}^{n_2}$, and the last n_1 are $\bar{r}^{n_1}$. Then for each x in $S'=[c,d]$,

$$\equiv H^{n_1}(H^{n_2}(d,\underline{r}^{n_2}),\bar{r}^{n_1})$$

$$\leqq H^{n_1}(d,\bar{r}^{n_1})$$

$$< \frac{a+b}{2}.$$

Similarly, if we define $\underline{r}^{n_1+n_2} = (\bar{r}^{n_1},\underline{r}^{n_2}) \in \mathcal{E}^{n_1+n_2}$ then $H^{n_1+n_2}(x,\underline{r}^{n_1+n_2}) > a+b/2$ for each $x \in S'$. Putting $m = n_1 + n_2$ proves (50) and concludes proof of Theorem 10.

QED

Proof of Theorem 11: From Lemma 6 there is an $\varepsilon > 0$ such that for all r, $H(x,r) > x$ for all x in $(0,\varepsilon)$, so we may redefine our state space to be $S' = S - [0,\varepsilon)$. We proceed to show that $[x_m,x_M]$ is the unique minimal γ-invariant closed interval in S', so that by applying Theorem 10 we may conclude the proof of this theorem. First, $[x_m,x_M]$ is a closed γ-invariant set, since for $x \in [x_m,x_M]$, $x_m = H_m(x_m) \leqq H_m(x) \leqq H(x,r) \leqq H_M(x) \leqq H_M(x_M) = x_M$, so γ a.e., $H(x,r) \in [x_m,x_M]$. Next, we show that $[x_m,x_M]$ is a minimal closed γ-invariant set. Suppose $[a,b] \subset [x_m,x_M]$ is γ-invariant. If we assume $x_m < a$, then $[a,b]$ γ-invariant implies $H(a,r) \geqq a$, so $H_m(a) \geqq a$, which contradicts Lemma 7(c). Hence, $x_m = a$. Similarly, we may show $b = x_M$, so $[x_m,x_M]$ is a minimal closed γ-invariant set.

Finally, we show that $[x_m,x_M]$ is the only minimal closed γ-invariant subset of S'. Suppose $[a,b]$ with $b \leqq x_m$ is γ-invariant. Then $H(b,r) \leqq b$ so $H_M(b) \leqq b$, which, since $b \leqq x_m < x_M$, contradicts Lemma 7(d). Similarly, we may show that $[a,b]$ with $a \geqq x_M$ cannot be γ-invariant. Hence, $[x_n,x_M]$ is the unique minimal closed γ-invariant subset of S', hence applying Theorem 10 concludes proof.

QED

Proof of Theorem 12: Define $x_{m-} = \max\{x \in [y_m,x_M] | H_m(x) = x\}$, and proceed as in Lemma 7(a) to show that x_{m-} is well defined. From Lemma 6, there is an $\varepsilon > 0$ such that $H(x,r) > x$ for any x in $(0,\varepsilon)$ and for all r. Define $S'' = [\varepsilon,x_M]$, then one can check that S'' is γ-invariant. We may then follow the steps in the proof of Theorem 11 to show that $[x_{m-},x_M]$ is the only minimal γ-invariant closed interval in S''. An application of Theorem 10 then proves that if $x_0 \in (0,x_M]$, then $F_t(x)$ converges as $t \to \infty$ uniformly in x to an invariant distribution $\underline{F}(x)$ with support $[x_{m-},x_M]$.

Similarly, if we define $x_{M+} = \min\{x \in [x_m,y_M] | H_M(x) = x\}$, we may show that if $x_0 \in [x_m,\infty)$ then $F_t(x)$ converges as $t \to \infty$ uniformly in x to an invariant distribution $\bar{F}(x)$ with support $[x_m,x_{M+}]$.

QED

Proof of Theorem 13: From (T.10), with $z' = h(z)$, we obtain

$$\gamma(\{r \in \mathcal{E} | H(x,r) \leqq z' \text{ for each } x \text{ in } S\}) > 0$$

and

$$\gamma(\{r \in \mathcal{E} | H(x,r) \geqq z' \text{ for each } x \text{ in } S\}) > 0$$

So γ splits, and we may apply Theorem 9 to complete the proof of the theorem.

QED

Proof of Theorem 14

(a) Suppose $(\underline{x},\underline{c},\underline{y})$ and $(\underline{x}',\underline{c}',\underline{y}')$ are optimal processes from some initial stock $y > 0$. Define a new process $(\bar{\underline{x}},\bar{\underline{c}},\bar{\underline{y}})$ as follows: Fix any α in $(0,1)$. Let $\bar{x}_t \equiv \alpha x_t + (1-\alpha)x'_t$ for $t \geqq 0$; $\bar{y}_0 \equiv y$ and for $t \geqq 1$, $\bar{y}_t \equiv f(\bar{x}_{t-1}, r_t)$; and for $t \geqq 0$, $\bar{c}_t \equiv \bar{y}_t - \bar{x}_t$. Clearly, $\bar{x}_t \geqq 0$ and $\bar{y}_t \geqq 0$ for each $t \geqq 0$. Further, using (T.11),

$$\bar{c}_t = f(\bar{x}_{t-1}, r_t) - \bar{x}_t = f(\alpha x_{t-1} + (1-\alpha)x'_{t-1}, r_t) - \alpha x_t - (1-\alpha)x'_t$$

$$\geqq \alpha f(x_{t-1}, r_t) + (1-\alpha) f(x'_{t-1}, r_t) - \alpha x_t - (1-\alpha)x'_t \tag{59}$$

$$= \alpha c_t + (1-\alpha)c'_t \tag{60}$$

$$\geqq 0$$

Hence, $(\bar{\underline{x}},\bar{\underline{c}},\bar{\underline{y}})$ is a well-defined (resp. input, consumption and stock) process from initial stock y. Under (U.2), the monotonicity of u, we obtain from (60),

$$Eu(\bar{c}_t) \geqq Eu(\alpha c_t + (1-\alpha)c'_t) \text{ for each } t \geqq 0 \tag{61}$$

If for some $\tau \geqq 1$, $x_{\tau-1}$ is different from $x'_{\tau-1}$ on a set with strictly positive probability, then under (T.11) (the *strict* concavity of f), (59) above holds with strict inequality; hence, (60) holds with strict inequality for $t = \tau$, so under (U.2), the strict monotonicity of u,

$$Eu(\bar{c}_\tau) > Eu(\alpha c_\tau + (1-\alpha)c'_\tau) \tag{62}$$

Then, using (61) and (62),

$$\sum_{t=0}^{\infty} \delta^t Eu(\bar{c}_t) > \sum_{t=0}^{\infty} \delta^t Eu(\alpha c_t + (1-\alpha)c'_t)$$

$$\geqq \alpha \sum_{t=0}^{\infty} \delta^t Eu(c_t) + (1-\alpha) \sum_{t=0}^{\infty} \delta^t Eu(c'_t) \text{ [from (U.3')]} \tag{63}$$

$$= \alpha V(y) + (1-\alpha) V(y) \text{ [Since } (\underline{x},\underline{c},\underline{y}), (\underline{x}',\underline{c}',\underline{y}') \text{ are optimal]}$$

$$= V(y)$$

This is a contradiction, hence proves $x_t = x'_t$ a.e. for each $t \rightarrow \geqq 0$. From this one obtains immediately that $y_t = y'_t$ and $c_t = c'_t$ a.e. for each $t \geqq 0$.

(b) This follows immediately from (a) above.

(c) Fix any y, y' and suppose $y > y' \geqq 0$. Let $(\underline{x},\underline{c},\underline{y})$ and $(\underline{x}',\underline{c}',\underline{y}')$ be optimal processes from initial stocks y and y', respectively. Fix any α in $(0,1)$. Define a new process $(\underline{\bar{x}},\underline{\bar{c}},\underline{\bar{y}})$ as follows: For all $t \geqq 0$, $\bar{x}_t \equiv \alpha x_t + (1-\alpha)x'_t$; $\bar{y}_0 \equiv \alpha y + (1-\alpha)y'$ and for $t \geqq 1$, $\bar{y}_t \equiv f(\bar{x}_{t-1}, r_t)$; and for all $t \geqq 0$, $\bar{c}_t \equiv \bar{y}_t - \bar{x}_t$. One employs an argument used in (a) above to show that $(\underline{\bar{x}},\underline{\bar{c}},\underline{\bar{y}})$ is a well-defined (resp. input, consumption and stock) process from $\bar{y}_0 \equiv \alpha y + (1-\alpha)y'$; and,

$$Eu(\bar{c}_t) \geqq Eu(\alpha c_t + (1-\alpha)c'_t) \text{ for each } t \geqq 0. \tag{64}$$

Then, $V(\alpha y + (1-\alpha)y') = V(\bar{y}_0) \geqq \sum_{t=0}^{\infty} \delta^t Eu(\bar{c}_t)$

$$\geqq \sum_{t=0}^{\infty} \delta^t Eu(\alpha c_t + (1-\alpha)c'_t) \text{ [from (64)]}$$

$$\geqq \alpha \sum_{t=0}^{\infty} \delta^t Eu(c_t) + (1-\alpha) \sum_{t=0}^{\infty} \delta^t Eu(c'_t) \text{ [from (U.3')]}$$

$$= \alpha V(y) + (1-\alpha) V(y') \tag{65}$$

Hence, V is concave.

(d) To show that V is strictly concave under (U.3), we modify the proof in (c) above as follows: Let y, y', α, $(\underline{x},\underline{c},\underline{y})$, $(\underline{x}',\underline{c}',\underline{y}')$ and $(\underline{\bar{x}},\underline{\bar{c}},\underline{\bar{y}})$ be as in (c) above. If at some date $\tau \geqq 0$, c^τ is different from c'_τ on a set with strictly positive probability, then (64) and (U.3) imply,

$$Eu(\bar{c}_\tau) \geqq Eu(\alpha c_\tau + (1-\alpha)c'_\tau) > \alpha Eu(c_\tau) + (1-\alpha) Eu(c'_\tau) \tag{66}$$

If, alternatively, $c_t = c'_t$ for each $t \geqq 0$, then since $y > y'$, we have $x_0 > x'_0$. So one may mimic the argument used to obtain (62) above, to show that

Finally, repeating the arguments used to obtain (65), with the aid of (66) and (67), we obtain,

$$V(\alpha y+(1-\alpha)y')>\alpha V(y)+(1-\alpha)V(y') \tag{68}$$

Hence, V is strictly concave.

QED

Proof of Theorem 15

(a) From Theorem 14(b), $\varphi(y)$ is single valued, hence $h(y)=\bar{h}(y)=\underline{h}(y)$ for each $y>0$ (where $\bar{h}$, $\underline{h}$ are defined in (12)). The monotonicity and continuity of h then follows from Theorem 3(b) and (c).

(b) Since $c(y)=y-h(y)$, the continuity of $c(y)$ follows from the continuity of $h(y)$, which has been established in (a) above.

To prove the monotonicity of $c(y)$ we use a method similar to that used in Theorem 2. Let $y>y'>0$ and suppose on the contrary that $c_0\equiv c(y)<c(y')\equiv c_0'$. Define $x_0\equiv h(y)$ and $x_0'\equiv h(y')$.

Let $\bar{x}_0\equiv y-c_0'$. Then $y\geqq y-c_0'\equiv\bar{x}_0$ and $\bar{x}_0\equiv y-c_0'>y'-c_0'=x_0'\geqq 0$. Hence, $\bar{x}_0\in[0,y]$. From Theorem 14(a), the optimal process from y is unique. However, $\bar{x}_0\equiv y-c_0'<y-c_0=x_0$, and x_0 is the unique optimal input from y, hence $\bar{x}_0$ is not an optimal input from y. Then, using functional equation (see (10) and (11) above),

$$u(c_0)+M(x_0)=V(y)>u(y-\bar{x}_0)+M(\bar{x}_0) \tag{69}$$

where

$$M(x)\equiv\delta\int V(f(x,r))\gamma(dr) \tag{70}$$

Next, let $\bar{x}_0'\equiv y'-c_0$. Then $y'\geqq y'-c_0\equiv\bar{x}_0'$ and $\bar{x}_0'\equiv y'-c_0>y'-c_0'=x_0'\geqq 0$. Hence, $\bar{x}_0'\in[0,y']$. So using the functional equation again we obtain

$$u(c_0')+M(x_0')=V(y')\geqq u(y'-\bar{x}_0')+M(\bar{x}_0') \tag{71}$$

Adding (69) and (71), and noting that $y-\bar{x}_0=c_0'$ and $y'-\bar{x}_0'=c_0$, we obtain

$$M(x_0)+M(x_0')>M(\bar{x}_0)+M(\bar{x}_0') \tag{72}$$

Notice that $\bar{x}_0\equiv y-c_0'<y-c_0=x_0$ and $\bar{x}_0\equiv y-c_0'>y'-c_0'=x_0'$.

Hence, there is a θ in $(0,1)$ such that $\bar{x}_0=\theta x_0+(1-\theta)x_0'$. Then $\bar{x}_0'\equiv y'-c_0=(y-c_0)+(y'-c_0')-(y-c_0')=x_0+x_0'-\bar{x}_c=(1-\theta)x_0+\theta x_0'$. Under (T.11) and the concavity of V (Theorem 14(c)), $M(x)$ is concave. Hence,

$$M(\bar{x}_0) \geqq \theta M(x_0) + (1-\theta)M(x_0') \tag{73}$$

$$M(\bar{x}_0') \geqq (1-\theta)M(x_0) + \theta M(x_0') \tag{74}$$

so by addition,

$$M(\bar{x}_0) + M(\bar{x}_0') \geqq M(x_0) + M(x_0') \tag{75}$$

This contradicts (72) and concludes proof of the theorem.

QED

Proof of Theorem 16

(a) This follows immediately from Theorem 14(c) and Theorem 6.

(b) This follows immediately from (a) above and Theorem 8.

Proof of Corollary 16

(a) This is Corollary 5!

(b) Let $y > y' > 0$. From Theorem 14(d), $V(\cdot)$ is strictly concave, hence $V'(\cdot)$ is strictly decreasing. Then, using Theorem 16(b), $u'(c(y)) = V'(y) < V'(y') = u'(c(y'))$; so under (U.3), the strict concavity of u, $c(y) > c(y')$ for $y > y' > 0$. Finally, if $y > y' = 0$, then $c(y') = 0$ and, from Theorem 4, $c(y) > 0$, so $c(y) > c(y')$. Hence, $c(y) > c(y')$ for all $y > y' \geqq 0$.

QED

Proof of Theorem 17: Let x_1, x_2 be any fixed points of H_m, h_M, respectively. We will show that $x_1 \leqq x_2$, from which the conclusion follows.

Since $\mathcal{E}$ is finite, there exists r_m, r_M in $\mathcal{E}$ such that

$$x_1 = H_m(x_1) = H(x_1, r_m) \text{ and } x_2 = H_M(x_2) = H(x_2, r_M) \tag{76}$$

From the stochastic Ramsey–Euler condition (Theorem 5),

$$u'(c(f(x_1, r_m))) = \delta \int u'(c(f(H(x_1, r_m), \sigma)) f'(H(x_1, r_m), \sigma)\gamma(d\sigma)$$

$$= \delta \int u'(c(f(x_1, \sigma))) f'(x_1, \sigma)\gamma(d\sigma) \tag{77}$$

But, $f(x_1, r_m) = f_m(x_1) \leqq f(x_1, \sigma)$ for each σ in $\mathcal{E}$, so since $c(y)$ is monotone non-decreasing (Theorem 6), and u is concave, we obtain $u'(c(f(x_1, r_m))) \geqq u'(c(f(x_1, \sigma)))$ for each σ in $\mathcal{E}$. Putting this in (77) gives

$$1 \leqq \delta \int f'(x_1, \sigma)\gamma(d\sigma) \tag{78}$$

... that $1 > \delta \int f'(x, \sigma)\gamma(d\sigma)$; so combining with (78)

$$\int f'(x_1,\sigma)\gamma(d\sigma) \geqq \int f'(x_2,\sigma)\gamma(d\sigma) \tag{79}$$

Strict concavity of $f(\cdot,\sigma)$ for each σ implies that the function $\int f'(x,\sigma)\gamma(d\sigma)$ is decreasing in x, so from (79) we obtain $x_1 \leqq x_2$.

QED

Proof of Theorem 18: We shall show that for each $y>0$, $\underline{h}(y)>0$ (where $\underline{h}$ is defined in (12)). This will conclude the proof of the theorem; for, suppose $(\underline{x},\underline{c},\underline{y})$ is any optimal process from a given initial stock $y>0$. Then by definition of $\underline{h}$, $x_0 \geqq \underline{h}(y)$; then using (T.5), $y_1 = f(x_0,r)>0$ for each r; hence $x_1 \geqq \underline{h}(y_1) = \underline{h}(f(x_0,r))>0$ for each r. Repeating such an argument, we obtain that for each date $t \geqq 0$, $x_t>0$ and $y_t>0$ a.s.

To prove $\underline{h}(y)>0$, for all $y>0$, fix $y>0$ and let $(\underline{x},\underline{c},\underline{y})$ be the optimal process from initial stock y, generated by the optimal investment policy function $\underline{h}$, and suppose on the contrary that $x_0=0$ and $c_0=y$. Then under (T.5), $x_t=0$, $y_t=0$ and $c_t=0$ a.s. for all $t \geqq 1$. Fix any ε in $(0,y)$. Then define a new process $(\underline{x}',\underline{c}',\underline{y}')$ as follows: $y_0' \equiv y$, $x_0' \equiv \varepsilon$, $c_0' \equiv y-\varepsilon$, $y_1' \equiv f(x_0',r) = f(\varepsilon,r)$, $x_1' \equiv 0$, $c_1' \equiv y_1' = f(\varepsilon,r)$ and for all $t \geqq 2$, $c_t' = x_t' = y_t' = 0$ a.s. Clearly, $(\underline{x}',\underline{c}',\underline{y}')$ is a well-defined (resp. input, consumption and stock) process from y.

Let $\tilde{V}(y)$ be the expected discounted total utility of the process $(\underline{x}',\underline{c}',\underline{y}')$. Then since $(\underline{x},\underline{c},\underline{y})$ is an optimal process from y,

$$0 \leqq V(y) - \tilde{V}(y) = \sum_{t=0}^{\infty} \delta^t E[u(c_t) - u(c_t')]$$

$$= u(y) - u(y-\varepsilon) + \delta \int [u(0) - u(f(\varepsilon,r))]\gamma(dr)$$

Hence,

$$\int [u(f(\varepsilon,r)) - u(0)]\gamma(dr) \leqq \frac{1}{\delta}[u(y) - u(y-\varepsilon)] \tag{80}$$

Define

$$A(\varepsilon,r) = \frac{u(f(\varepsilon,r)) - u(f(0,r))}{f(\varepsilon,r) - f(0,r)}$$

and $B(\varepsilon,r) = [f(\varepsilon,r) - f(0,r)]/\varepsilon$.

Then using Fatou's lemma (see, e.g., Chung (1974), p.42),

$$\int \liminf_{\varepsilon\to 0} A(\varepsilon,r) \cdot \liminf_{\varepsilon\to 0} B(\varepsilon,r)\gamma(dr) \leqq \int \liminf_{\varepsilon\to 0} A(\varepsilon,r)B(\varepsilon,r)\gamma(dr)$$

$$=\liminf_{\varepsilon\to 0}\int\frac{u(f(\varepsilon,r))-u(f(0,r))}{\varepsilon}\gamma(dr)$$

$$\leqq\liminf_{\varepsilon\to 0}\frac{1}{\delta}\left[\frac{u(y)-u(y-\varepsilon)}{\varepsilon}\right][\text{from}(80)]$$

$$=\frac{1}{\delta}u'(y)<\infty\ [\text{from (U.4)}] \tag{81}$$

This is a contradiction since the left-hand side of (81) is infinite under (U.5) and (T.12).

QED

Proof of Theorem 19: From Theorem 18 above, we obtain that optimal input and stock processes are interior (notice that (T.11) and (T.3) imply (T.12)). It remains to show that optimal consumption processes are interior. It suffices to show that $c(y)>0$ for each $y>0$, where $c(\cdot)$ is the optimal consumption policy function; for suppose $c(y)>0$ for each $y>0$. Then if $(\underset{\sim}{x},\underset{\sim}{c},\underset{\sim}{y})$ is an optimal process from some fixed initial stock $y>0$, we obtain from Theorem 18 that $y_t>0$ a.s., hence $c_t=c(y_t)>0$ a.s. for each $t\geqq 0$.

To show $c(y)>0$ for each $y>0$, fix $y>0$; let $(\underset{\sim}{x},\underset{\sim}{c},\underset{\sim}{y})$ be the optimal process from initial stock y, and suppose on the contrary that $c_0=0$ and $x_0=y$. From the functional equation (see (10) and (11)), for all $0<\varepsilon<\frac{1}{2}y$,

$$u(0)+\delta\int V(f(y,r))\gamma(dr)=V(y)\geqq u(\varepsilon)+\delta\int V(f(y-\varepsilon,r))\gamma(dr)$$

hence

$$\frac{u(\varepsilon)-u(0)}{\varepsilon}\leqq\delta\int\frac{V(f(y,r))-V(f(y-\varepsilon,r))}{f(y,r)-f(y-\varepsilon,r)}\cdot\frac{f(y,r)-f(y-\varepsilon,r)}{\varepsilon}\gamma(dr) \tag{82}$$

Recall $f_m(x)\equiv\min_r f(x,r)$ and $f_M(x)\equiv\max_r f(x,r)$. Since the value function, V, is concave (Theorem 14(c)), and $f(y,r)\geqq f(y-\varepsilon,r)\geqq f_m(\frac{1}{2}y)$,

$$\frac{V(f(y,r))-V(f(y-\varepsilon,r))}{f(y,r)-f(y-\varepsilon,r)}\leqq\frac{V(f_m(y/2))-V(0)}{f_m(y/2)} \tag{83}$$

Next, since the production function is concave (T.11),

$$\frac{f(y,r)-f(y-\varepsilon,r)}{\varepsilon}\leqq\frac{f(y/2,r)}{y/2}$$

for each r in $\mathcal{E}$; also,

Putting this and (83) into (82) gives,

$$\frac{u(\varepsilon)-u(0)}{\varepsilon} \leqq \delta \frac{V(f_m(y/2))-V(0)}{f_m(y/2)} \cdot \frac{f_M(y/2)}{(y/2)} < \infty \tag{84}$$

Taking limits as $\varepsilon \to 0$ in (84) yields a contradiction since the left-hand side tends to infinity under (U.5), while the right-hand side is finite.

QED

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