JOURNAL OF ECONOMIC THEORY 45, 316–329 (1988)

On Characterizing Optimality of Stochastic Competitive Processes

YAW NYARKO*

Department of Economics, Brown University, Providence, Rhode Island 02912

Received August 20, 1986; revised August 20, 1987

A condition is provided to replace the transversality condition in characterizing the optimality of competitive processes. This extends the work of W. Brock, L. Hurwicz, and M. Majumdar ("On Characterizing Optimality of Competitive Programs in Terms of Decentralizable Conditions" and "Optimal Intertemporal Allocation Mechanisms and Decentralization of Decisions," Cornell University Working Paper Nos. 333 and 369 (1985)) on providing an informationally decentralizable condition for characterizing optimality in stochastic infinite-horizon models without discounting. *Journal of Economic Literature* Classification Numbers: 026, 011.

I. INTRODUCTION

In the literature on the theory of intertemporal resource allocation it is known that optimality for infinite-horizon economies can be characterized in terms of two conditions; the first requires intertemporal profit maximization and utility maximization relative to a system of "competitive" prices, while the second is a transversality condition, which requires (for undiscounted models) that the value of the capital stock, computed at the competitive prices, be uniformly bounded over time. These results, for stochastic economies, have been proved in Zilcha [11, 12].

There are two problems with the above characterization of optimality. The first is that the transversality condition involves a limit, so one cannot verify on a period-by-period basis whether or not this condition is being attained. I shall call this the absence of *temporal decentralization.* The second problem is the absence of *informational decentralization,* in the sense of Hurwicz [8, Definition 10, p. 401]. In particular, one cannot design a

* I thank Professor Tapan Mitra for his valuable assistance. This paper is a revision of Chapter 4 of my Ph.D. thesis submitted to Cornell University, and supervised by Professor M. Majumdar. Research reported here was supported by the National Science Foundation under Grant SES 8304131 awarded to Professor M. Majumdar.

0022-0531/88 $3.00

meaningful recource allocation mechanism that will ensure that the transversality condition will be met, but which is constrained so that the rules of behavior of agents at each date depend only upon the partial history at that date. One should consult [9] for futher motivation.

Brock and Majumdar [3] provide a condition to replace the transversality condition, which is both temporally and informationally decentralized. The purpose of this paper is to extend their result to handle stochastic economies; in particular to the model of intertemporal resource allocation under uncertainty discussed in [10].

The main result of this paper may be described as follows: let $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{c}})$ be the optimal stationary process and $\{\hat{p}_t\}$ the corresponding competitive (or supporting) price process (see (3.9) and (3.10)). Then a resource allocation process $(\mathbf{x}, \mathbf{y}, \mathbf{c})$ is optimal if and only if (a) the process is competitive (see (3.7) and (3.8)) at some prices $\{p_t\}$ and (b) at each date t, $E(p_t - \hat{p}_t)(y_t - \hat{y}_t) \leqslant 0$.

The rest of this paper is organized as follows: In Section II some preliminary notation is introduced and in Section III the model is formally presented. The main results are stated in Section IV.

II. Some Preliminary Notation

R^n is the n-dimensional Euclidean space. Given any two vectors $a = (a_1, ..., a_n)$ and $b = (b_1, ..., n_n)$ in R^n we write $a \geqslant b$ if $a_i \geqslant b_i$ for each $i = 1, ..., n$; $a > b$ if $a \geqslant b$ and a is different from b; and $a \gg b$ if $a_i > b_i$ for each $i = 1, ..., n$. $R^n_+ = \{x \in R^n : x \geqslant 0\}$ and $R^n_{++} = \{x \in R^n : x \gg 0\}$. We denote by $\|\cdot\|$ the "max" norm in R^n; i.e., if $a = (a_1, ..., a_n) \in R^n$, then $\|a\| = \max\{|a_1|, ..., |a_n|\}$.

III. The Model

The framework we use is essentially the model of intertemporal resource allocation with production uncertainty, where labor is necessary for production, studied in [10, 4, 11, 12]. We shall, however, consider the version of the model that has been reduced to per capita terms. In particular the assumptions we place on the environment[1] and the technology are similar to those in [10, 4] stated in per capita terms.

[1] The assumptions we place on the environment are similar to those in [4], which is a generalization of [10]. In particular, [10] assumes that the set of states of the environment at any date is finite, the probability measure on the sequence of states, σ, is atomless, and also that the shift operator, T, is ergodic. These are used in [10] to obtain bounds on production processes. Following [4] we impose these bounds directly in assumptions (T7) and (T.8) of this paper.

III.a. *The Environment*

The environment is represented by the probability space (S, ϕ, σ), where

(i) S is the set of doubly-infinite sequences $s = (s_t)$, $-\infty < t < \infty$, with s_t representing the state of the environment at date t, which lies in some complete and separable metric space E. In particular, $S = X_{t=-\infty}^{\infty} E_t$, where for each t, $E_t = E$.

(ii) ϕ is the sigma field generated by cylinder sets in S (i.e., generated by sets of the form $X_{t=-\infty}^{\infty} B_t$, where $B_t = E$ for all but finitely many t, and where B_t belongs to the Borel sigma field on E for all t).

(iii) σ is a probability measure on ϕ, the probability distribution on the sequences of states.

The shift operator $T: S \to S$ is defined by

$$(Ts)_t = s_{t+1}. \tag{3.1}$$

We assume

(E.1) T is measure preserving.[2, 3]

Assumption (E.1) means that if A is any set in ϕ, then $\sigma(A) = \sigma(TA)$. For any integer i, $-\infty < i < \infty$, T^i is the ith iterate of T, i.e., $(T^i s)_t = s_{t+i}$. Given any function f on S we define $T^i f$ by

$$T^i f(s) = f(T^i s). \tag{3.2}$$

Let ϕ_t denote the sigma field generated by the partial history $(..., s_{t-1}, s_t)$; ϕ_t is the sigma field generated by cylinder sets $X_{i=-\infty}^{\infty} B_i$, where $B_i = E$ for all $i > t$.

Recall that $\|\cdot\|$ denotes the norm in R^n (defined in Section II). We define for each $t = 0, 1, ...,$

$$L^1(\phi_t) = \left\{ f\colon S \to R^n \mid f \text{ is } \phi_t\text{-measurable and } \int \|f(s)\| \, d\sigma < \infty \right\} \tag{3.3}$$

$$L^\infty(\phi_t) = \{ f\colon S \to R^n \mid f \text{ is } \phi_t\text{-measurable and essentially bounded}^4 \}. \tag{3.4}$$

[2] Assumption (E.1) implies that T^{-1} is measure preserving.

[3] One consequence of the assumption that T is measure preserving is the following: Given any ϕ-measurable function f, $\int f(s)\, d\sigma = \int f(Ts)\, d\sigma$. One should consult [1] for more on measure-preserving operators.

[4] f is essentially bounded if there is a $0 < C < \infty$ such that $\|f(s)\| \leq C$ a.s.

III.b. *The Technology*

The technology is described by the correspondence $\tau: R^n_+ \times S \to R^n_+$; $\tau(x, s)$ is the set of output possibilities at date 1 if the input is $x \in R^n_+$ at date 0 and the state of environment is $s \in S$. Define $B(s) = \{(x, y) \in R^n_+ \times R^n_+ \mid y \in \tau(x, s)\}$ for each $s \in S$. We impose the following assumptions on the technology:

(T.1) For all $s \in S$, $B(s)$ is closed and convex and contains (0, 0).

(T.2) For all $s \in S$, "$(x, y) \in B(s)$, $x' \geqslant x$, and $y' \leqslant y$" implies "$(x', y') \in B(s)$" [free disposal].

(T.3) For all $s \in S$, if $(0, y) \in B(s)$ then $y = 0$ [importance of inputs].

(T.4) For all $x \in R^n_+$, $\tau(x, \cdot)$ is ϕ_1-measurable (i.e., for any closed set F in R^n_+, $\{s \mid \tau(x, s) \cap F \text{ is not empty}\}$ is in ϕ_1) [measurability].

Assumption (T.4) is the requirement that production possibilities at date 1 depend (measurably) on the history of the environment up to date 1.

Define $G = \{(\alpha, \beta) \mid \alpha,\ T^{-1}\beta \in L^\infty(\phi_0) \text{ and } (\alpha(s), \beta(s)) \in B(s) \text{ a.s.}\}$, the set of all (stationary) production plans. Any (α, β) in G has the following interpretation: $\alpha(s)$ is the input at date 0 and $\beta(s)$ is the corresponding output at date 1 when the state is $s \in S$. We assume that there is a production plan whose net output is bounded away from zero:

(T.5) There is a $(\bar{\alpha}, \bar{\beta}) \in G$ such that $\bar{\beta}(T^{-1}s) - \bar{\alpha}(s) \geqslant \bar{c}$ a.s. for some $\bar{c} \in R^n_{++}$.

We also require the following convexity assumption on the G:

(T.6) For any (α, β) and (α', β') in G with $\alpha(s)$ different from $\alpha'(s)$ on a set D with positive measure, and for any ϕ_0-measurable random variable a, with $0 < a(s) < 1$ a.s., there exists a β'' in $L^\infty(\phi_1)$ such that $\beta''(s) \geqslant a(s)\, \beta(s) + (1 - a(s))\, \beta'(s)$ a.s. with strict inequality for any s in D, and $(a\alpha + (1 - a)\, \alpha', \beta'') \in G$ [weak strict convexity of outputs].

A *feasible process* from initial stock $y \in L^\infty(\phi_0)$ is a stochastic process $\{x_t, y_t, c_t\}_{t=0}^\infty$ which satisfies:

(a) $(x_t, y_{t+1}) \in T^t G$ for $t = 0, 1, 2, \ldots$;

(b) $y_0 = y$ and $c_t = y_t - x_t \geqslant 0$ a.s. for $t = 0, 1, 2, \ldots$,

where $T^t G = \{(\alpha_t, \beta_t): \alpha_t = T^t\alpha \text{ and } \beta_t = T^t\beta \text{ for some } (\alpha, \beta) \in G\}$. (The shift operator, T, and its tth iterate, T^t, are defined in Section III.a.)

Let $P(y)$ be the set of all feasible processes from initial stock $y \in L^\infty(\phi_0)$. We shall denote a feasible process by $(\mathbf{x}, \mathbf{y}, \mathbf{c})$, where $\mathbf{x} = \{x_t\}_{t=0}^\infty$, $\mathbf{y} = \{y_t\}_{t=0}^\infty$, and $\mathbf{c} = \{c_t\}_{t=0}^\infty$; we refer to $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{c}$ as the input, output, and consumption processes, respectively.

Let $G_0 = \{(\alpha, \beta) \in G | \beta(T^{-1}s) - \alpha(s) \geqslant 0$ a.s.$\}$, the set of all feasible stationary production programs; each (α, β) in G_0 defines a stationary feasible process $(\mathbf{x}, \mathbf{y}, \mathbf{c})$, where for each $t \geqslant 0$ and $s \in S$,

$$x_t(s) = \alpha(T^t s), \qquad y_t(s) = \beta(T^{t-1}s), \qquad c_t(s) = y_t(s) - x_t(s).$$

The next two assumptions place bounds on feasible processes and stationary production programs. Recall that $\|\cdot\|$ is the norm of R^n_+ (defined in Section II).

(T.7) For each $y \in L^\infty(\phi_0)$ there is a $0 < K(y) < \infty$, such that if $(\mathbf{x}, \mathbf{y}, \mathbf{c}) \in P(y)$, then for all $t \geqslant 0$, $\|x_t(s)\| \leqslant K(y)$ a.s., $\|y_t(s)\| \leqslant K(y)$ a.s., and $\|c_t(s)\| \leqslant K(y)$ a.s.

(T.8) There is a $0 < Q < \infty$ such that for all $(\alpha, \beta) \in G_0$, $\|\alpha(s)\| \leqslant Q$ a.s. and $\|\beta(s)\| \leqslant Q$ a.s.

Remark. Note that the bound in (T.7) clearly depends on the initial stock. The bounds in (T.7) and (T.8) may be obtained by starting with the model in [10], where[1] labor is necessary for production, and then assuming that the labor supply process is bounded and finally reducing to per capita units (see [10, Theorem 3.1 and Lemma 3.2]).

III.c. *Optimality and Competitive Prices*

Feasible processes are evaluated according to the utilities generated by the corresponding consumption process. Let u: $R^n_+ \to R$ be the one-period utility function. We assume

(U.1) u is continuous on R^n_+ ;

(U.2) u is strictly increasing (i.e., $c_1 > c_2$ implies $u(c_1) > u(c_2)$);

(U.3) u is strictly concave on R^n_+.

A process $(\mathbf{x}^*, \mathbf{y}^*, \mathbf{c}^*)$ in $P(y)$ is *optimal* if for all $(\mathbf{x}, \mathbf{y}, \mathbf{c})$ in $P(y)$

$$\limsup_{N \to \infty} \sum_{t=0}^{N} [Eu(c_t) - Eu(c_t^*)] \leqslant 0, \tag{3.5}$$

where E is the expectation operator.

A program $(\hat{\alpha}, \hat{\beta}) \in G_0$ is said to be an *optimal stationary program* if

$$\sup_{(\alpha, \beta) \in G_0} \int u(\beta(T^{-1}(s)) - \alpha(s))\, d\sigma = \int u(\hat{\beta}(T^{-1}(s)) - \hat{\alpha}(s))\, d\sigma. \tag{3.6}$$

The existence of an optimal stationary program was first proved in [10].

LEMMA 3.1. (Radner). *There exists an optimal stationary program.*

Proof. See[1] [4, Corollary IV.1, p. 189]. ∎

Using the strict convexity assumption on the technology (T.6), in addition to the strict monotonicity and strict concavity assumptions on the utility function, (U.2) and (U.3), it is easy to show that the optimal stationary program, which we denote by $(\hat{x}, \hat{y})$, is unique. We define the *optimal stationary process*, $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{c}})$, by $\hat{x}_t(s) = \hat{x}(T^t s)$, $\hat{y}_t(s) = \hat{y}(T^t s)$, and $\hat{c}_t(s) = \hat{y}_t(s) - \hat{x}_t(s)$ for all $s \in S$ and $t = 0, 1, \ldots$.

A feasible process $(\mathbf{x}, \mathbf{y}, \mathbf{c})$ is a *competitive process* if there is a sequence $\{p_t\}_{t=0}^{\infty}$ such that for each t, $p_t \in L^1(\phi_t)$, $p_t(s) > 0$ a.s., and

$$u(c_t(s)) - p_t(s)\, c_t(s) \geq u(c) - p_t(s)\, c \qquad \text{a.s.} \quad \text{for all } c \text{ in } R^n_+ \tag{3.7}$$

$$E[p_{t+1}(s)\, y_{t+1}(s) \mid \phi_t] - p_t(s)\, x_t(s) \geq E[p_{t+1}(s)\, \beta(s) \mid \phi_t] - p_t(s)\, \alpha(s)$$
$$\text{a.s.} \qquad \text{for all } (\alpha, \beta) \text{ in } T^t G. \tag{3.8}$$

Remark. The competitive conditions (3.7) and (3.8) are sometimes stated as

$$\int u(c_t(s))\, d\sigma - \int p_t(s)\, c_t(s)\, d\sigma \geq \int u(\mu(s))\, d\sigma - \int p_t(s)\, \mu(s)\, d\sigma$$
$$\text{for all } \mu \in L^\infty(\phi_0) \text{ with } \mu(s) \geq 0 \quad \text{a.s.} \tag{3.7$'$}$$

$$\int p_{t+1}(s)\, y_{t+1}(s)\, d\sigma - \int p_t(s)\, x_t(s)\, d\sigma \geq \int p_{t+1}(s)\, \beta(s)\, d\sigma - \int p_t(s)\, \alpha(s)\, d\sigma$$
$$\text{for all } (\alpha, \beta) \text{ in } T^t G. \tag{3.8$'$}$$

As observed by [12, Remark on p. 519], since G is defined as the set of all measurable selections of $B(s)$, it is easy to show that (3.7) and (3.8) are equivalent to (3.7)′ and (3.8)′.

The existence of competitive prices for the optimal stationary process was first shown in [10]:

LEMMA 3.2 (Radner). *There exists a system of competitive prices, $\{\hat{p}_t\}$ for optimal stationary process, $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{c}})$. In particular, there is a $\hat{p} \in L^1(\phi_0)$ such that for all $t = 0, 1, \ldots, \hat{p}_t(s) = \hat{p}(T^t s) > 0$ a.s. and*

$$u(\hat{c}_t(s)) - \hat{p}_t(s)\hat{c}_t(s) \geq u(c) - \hat{p}_t(s)\, c \qquad \textit{a.s.} \quad \textit{for all } c \in R^n_+ \tag{3.9}$$

$$E[\hat{p}_{t+1}(s)\, \hat{y}_{t+1}(s) \mid \phi_t] - \hat{p}_t(s)\, \hat{x}_t(s) \geq E[\hat{p}_{t+1}(s)\, \beta(s) \mid \phi_t]$$
$$- \hat{p}_t(s)\, \alpha(s) \qquad \textit{a.s.} \quad \textit{for all } (\alpha, \beta) \textit{ in } T^t G. \tag{3.10}$$

Proof. See [4, Theorem VIII.1, p. 194]. ∎

Let $(\mathbf{x}, \mathbf{y}, \mathbf{c})$ be a competitive process with supporting prices $\{p_t\}$. We define

$$v_t = (p_t - \hat{p}_t)(x_t - \hat{x}_t). \tag{3.11}$$

In characterizing the optimality of competitive processes, v_t will play an essential role. An immediate consequence of the competitive conditions (3.7)–(3.10) is the following martingale result first noted by [6]:

LEMMA 3.3. *Let* $(\mathbf{x}, \mathbf{y}, \mathbf{c})$ *be a competitive process with supporting prices* $\{p_t\}$, *and let* v_t *be as in* (3.11). *Then* v_t *is a submartingale, and, in particular,* $Ev_t \geqslant Ev_0$ *for all* $t = 0, 1, \ldots$.

Proof. From the competitive conditions (3.8) and (3.10), for each $t \geqslant 0$,

$$E[p_{t+1}(y_{t+1} - \hat{y}_{t+1}) \mid \phi_t] \geqslant p_t(x_t - \hat{x}_t) \quad \text{a.s.} \tag{3.12}$$

$$E[\hat{p}_{t+1}(\hat{y}_{t+1} - y_{t+1}) \mid \phi_t] \geqslant \hat{p}_t(\hat{x}_t - x_t) \quad \text{a.s.} \tag{3.13}$$

so by addition,

$$E[(p_{t+1} - \hat{p}_{t+1})(y_{t+1} - \hat{y}_{t+1}) \mid \phi_t] \geqslant (p_t - \hat{p}_t)(x_t - \hat{x}_t) \quad \text{a.s.} \tag{3.14}$$

Next, from the competitive conditions (3.7) and (3.9),

$$u(c_t) - p_t c_t \geqslant u(\hat{c}_t) - p_t \hat{c}_t \quad \text{a.s} \tag{3.15}$$

$$u(\hat{c}_t) - \hat{p}_t \hat{c}_t \geqslant u(c_t) - \hat{p}_t c_t \quad \text{a.s} \tag{3.16}$$

so by addition,

$$(p_t - \hat{p}_t)(c_t - \hat{c}_t) \leqslant 0 \quad \text{a.s.} \tag{3.17}$$

Using (3.14) and (3.17) (for $t+1$) one obtains

$$E[(p_{t+t} - \hat{p}_{t+1})(x_{t+1} - \hat{x}_{t+1}) \mid \phi_t] \geqslant (p_t - \hat{p}_t)(x_t - \hat{x}_t) \quad \text{a.s.} \tag{3.18}$$

from which the lemma follows immediately. ∎

We refer to $\hat{x}$ and $\hat{y}$ as the *golden rule* input and output, resp. The golden rule input, $\hat{x}$, is said to be *expansible if for some* $z \in R^n_{++}$, $(\hat{x}, T\hat{x} + z) \in G$. Clearly, if the golden rule consumption, $\hat{c}(s) = \hat{y}(T^{-1}s) - \hat{x}(s)$, satisfies $\hat{c}(s) \geqslant \Gamma$ a.s. for some Γ in R^n_{++}, then $\hat{x}$ is necessarily expansible. We now show that if the golden rule input, $\hat{x}$, is expansible, then from $\hat{x}$ a multiple d (with $d > 1$) of $\hat{x}$ can be produced.

LEMMA 3.4. *Suppose that the golden rule input,* $\hat{x}$, *is expansible; then there is a* $d > 1$ *such that* $(\hat{x}, dT\hat{x}) \in G$.

Proof. Let $I=(1, ..., 1)\in R^n_{++}$. Since $\hat{x}$ is expansible, there is a $z=(z_1,..., z_n)\in R^n_{++}$ such that $(\hat{x}, T\hat{x}+z)\in G$. Let $\bar{z}=\min\{z_1, ..., z_n\}$; then $z\geqslant \bar{z}I\gg 0$. Also, from (T.8) there is a $0<Q<\infty$ such that $\|T\hat{x}(s)\|\leqslant Q$ a.s.; hence $0\leqslant T\hat{x}(s)\leqslant QI$ a.s. Let $d=1+(\bar{z}/2Q)$. Then $T\hat{x}(s)+z-dT\hat{x}(s)=(1-d)\,T\hat{x}(s)+z=-(\bar{z}/2Q)\,T\hat{x}(s)+z\geqslant -(\bar{z}/2)\,I+\bar{z}I=(\bar{z}/2)\,I\gg 0$. Hence $T\hat{x}(s)+z\gg dT\hat{x}(s)$ a.s.; but then since $(\hat{x}, T\hat{x}+z)\in G$, the free disposal assumption (T.2) implies that $(\hat{x}, dT\hat{x})\in G$. ∎

The following lemma will be used in proving the results in the next section, and may be of independent interest.

LEMMA 3.5. *Suppose that the golden rule input, $\hat{x}$, is expansible. Let $0<\bar{b}<1$. Then there is an $M\geqslant 1$ such that for all $b\geqslant\bar{b}$ there is a feasible process* $(\mathbf{x}, \mathbf{y}, \mathbf{c})$ *with $y_0=b\hat{x}$ and $x_t(s)=\hat{x}_t(s)$ a.s. for all $t\geqslant M$.*

Proof. Fix a $0<\bar{b}<1$ and let $d>1$ be as in Lemma 3.4 above. Since $\bar{b}d^t=\bar{b}<1$ for $t=0$ and $\bar{b}d^t\to\infty$ as $t\to\infty$, there is an $M'\geqslant 0$ such that

$$\bar{b}d^t\leqslant 1 \text{ for } t\leqslant M' \qquad \text{and} \qquad \bar{b}d^t>1 \text{ for } t>M'. \tag{3.19}$$

Let $M=M'+1$. Fix any $b\geqslant\bar{b}$. We now construct a process $(\mathbf{x}, \mathbf{y}, \mathbf{c})$ feasible from $b\hat{x}$. Define $y_0=b\hat{x}$, $x_0=\bar{b}\hat{x}$, and $c_0=y_0-x_0=(b-\bar{b})\,\hat{x}$; for $1\leqslant t\leqslant M'$, if there exists such a t, define $y_t=x_t=\bar{b}d^t\hat{x}_t$ and $c_t=0$; and for $t>M'$, define $y_t=x_t=\hat{x}_t$ and $c_t=0$.

To show that $(\mathbf{x}, \mathbf{y}, \mathbf{c})$ is feasible observe that $y_0=b\hat{x}$ and $c_t=y_t-x_t\geqslant 0$ for all $t\geqslant 0$, so it remains only to show that for all $t\geqslant 0$, $(x_t, y_{t+1})\in T^tG$.

First we show that

$$(\bar{b}d^t\hat{x}_t, \bar{b}d^t\, d\hat{x}_{t+1})\in T^tG \qquad \text{for all } 0\leqslant t\leqslant M'. \tag{3.20}$$

To see this note that from (3.19), for all $0\leqslant t\leqslant M'$, $\bar{b}d^t\leqslant 1$; but from Lemma 3.4 above, $(\hat{x}_t, d\hat{x}_{t+1})\in T^tG$, and from (T.1), $(0, 0)\in T^tG$, so (3.20) follows from the convexity assumption on the technology, (T.1).

We now use (3.20) to prove $(x_t, y_{t+1})\in T^tG$ for all $t=0, 1, ...$ If $0\leqslant t<M'$, then $(x_t, y_{t+1})=(\bar{b}d^t\hat{x}_t, \bar{b}d^t d\hat{x}_{t+1})$, so from (3.20), $(x_t, y_{t+1})\in T^tG$. If $t=M'$, we obtain from (3.19) that $\bar{b}d^{t+1}>1$ so $\bar{b}d^{t+1}\hat{x}_{t+1}(s)\geqslant\hat{x}_{t+1}(s)$ a.s.; but then (3.20) and the free disposal assumption (T.2) implies that $(x_t, y_{t+1})=(\bar{b}d^t\hat{x}_t, \hat{x}_{t+1})\in T^tG$. Finally, if $t>M'$, then $(x_t, y_{t+1})=(\hat{x}_t, \hat{x}_{t+1})$, which clearly belongs to T^tG.

With $M=M'+1$, it is easy to see that $(\mathbf{x}, \mathbf{y}, \mathbf{c})$ satisfies all the conclusions of the lemma. ∎

IV. The Main Theorems

In this section the principal results of this paper are proved. That is, we prove that optimality can be completely characterized in terms of period-by-period conditions (see (4.1) below), thereby replacing the transversality conditions.

THEOREM 4.1. *Suppose that the process* $(\mathbf{x}, \mathbf{y}, \mathbf{c})$ *is optimal from* $y_0 \gg 0$. *Then the process is competitive at some prices* $\{p_t\}$, *and*

$$E[(p_t - \hat{p}_t)(y_t - \hat{y}_t)] \leqslant 0 \qquad \textit{for all } t \geqslant 0. \tag{4.1}$$

Proof. From [11, Theorem 1, p. 434] we know that the optimal process is competitive at some system of prices $\{p_t\}$.

For any $y = 0, 1, \ldots$ and $y \in L^{\infty}(\phi_t)$, define

$$P_t(y) = \{\{x'_i, y'_i, c'_i\}_{i=t}^{\infty} \mid y'_t = y, (x'_i, y'_{i+1}) \in T^i G$$

and

$$c'_i = y'_i - x'_i \geqslant 0 \qquad \text{a.s.} \qquad \text{for all } i \geqslant t\}.$$

Next, define the function $g_t(\mathbf{x}', \mathbf{y}', \mathbf{c}')$ on $P_t(y)$ by

$$g_t(\mathbf{x}', \mathbf{y}', \mathbf{c}') = \liminf_{N \to \infty} \sum_{i=t}^{N} [Eu(c'_i) - Eu(\hat{c}_i)]$$

and

$$W_t(y) = \sup\{g_t(\mathbf{x}', \mathbf{y}', \mathbf{c}') \mid (\mathbf{x}', \mathbf{y}', \mathbf{c}') \in P_t(y)\}.$$

Then from the proof of [11, Theorem 1, p. 434], since $\{x_i, y_i, c_i\}_{i=t}^{\infty}$ and $\{\hat{x}_i, \hat{y}_i, \hat{c}_i\}_{i=t}^{\infty}$ are optimal[5] from y_t and $\hat{y}_t$, resp., we obtain

$$W_t(y_t) - W_t(\hat{y}_t) \geqslant E[p_t y_t - p_t \hat{y}_t] \tag{4.2}$$

$$W_t(\hat{y}_t) - W_t(y_t) \geqslant E[\hat{p}_t \hat{y}_t - \hat{p}_t y_t]. \tag{4.3}$$

Adding (4.2) and (4.3) and rearranging results in (4.1), the theorem is proved. ∎

[5] The assertion that the optimal stationary process is optimal from its own initial stock among the set of all feasible processes is not true without the stronger convexity assumption on the technology (assumption (T.6)), as an example in [2, p. 279] with linear technology indicates. The proof of the assertion under our assumptions is as follows:

THEOREM 4.2. *Suppose that the golden rule input, $\hat{x}$, is expansible. If* $(\mathbf{x}, \mathbf{y}, \mathbf{c})$ *is a competitive process at prices* $\{p_t\}$ *and* (4.1) *holds, then the process* $(\mathbf{x}, \mathbf{y}, \mathbf{c})$ *is optimal.*

Before we prove Theorem 4.2 we will state some preliminary lemmas which may be of independent interest. In each of these lemmas $(\mathbf{x}, \mathbf{y}, \mathbf{c})$ will be a competitive process and $\{p_t\}$ will be its supporting prices.

LEMMA 4.3. *Suppose that* (4.1) *holds. Then*

$$E[(p_t - \hat{p}_t)(x_t - \hat{x}_t)] \leqslant 0 \qquad \textit{for all } t \geqslant 0. \tag{4.1$'$}$$

Proof. This follows immediately from taking expectations in (3.14) and using (4.1). ∎

LEMMA 4.4. *Suppose that* (4.1) *holds. Then there there is a* $0 < H < \infty$ *such that*

$$Ep_t x_t - Ep_t \hat{x}_t \leqslant H \qquad \textit{for all } t \geqslant 0. \tag{4.4}$$

Proof. Fix a $t = 0, 1, \ldots$. Then using (4.1)$'$ in Lemma 4.3 above,

$$Ep_t(x_t - \hat{x}_t) \leqslant E\hat{p}_t(x_t - \hat{x}_t) \leqslant E\hat{p}_t x_t. \tag{4.5}$$

Let $(\mathbf{x}, \mathbf{y}, \mathbf{c})$ be any feasible process from initial stock $\hat{y}_0$. Then from the competitive condition (3.10), for all $N = 0, 1, \ldots,$

$$E \sum_{t=0}^{N} [u(c_t) - u(\hat{c}_t)] \leqslant E \sum_{t=0}^{N} [\hat{p}_t(c_t - \hat{c}_t)]$$

$$= E \sum_{t=0}^{N} [\hat{p}_t(y_t - \hat{y}_t) - \hat{p}_t(x_t - \hat{x}_t)]$$

$$= E \sum_{t=0}^{N} [\hat{p}_{t+1}(y_{t+1} - \hat{y}_{t+1}) - \hat{p}_t(x_t - \hat{x}_t)] - E\hat{p}_{N+1}(y_{N+1} - \hat{y}_{N+1})$$

$$+ \hat{p}_0(y_0 - \hat{y}_0).$$

From the competitive condition (3.10) the summation above is nonpositive; further, from [12, Corollary 1, p. 521], we may set $\lim_{N \to \infty} E\hat{p}_{N+1}(y_{N+1} - \hat{y}_{N+1}) = 0$. Hence, if $y_0 = \hat{y}_0$, taking limits in the expression above yields

$$\limsup_{N \to \infty} E \sum_{t=0}^{N} [u(c_t) - u(\hat{c}_t)] \leqslant 0,$$

which proves the assertion.

From assumption (T.7) there is a $0 < K(y_0) < \infty$ such that $\|x_t(s)\| \leqslant K(y_0)$ a.s. (where $\|\cdot\|$ is the norm on R^n_+, defined in Section II). Hence

$$E\hat{p}_t x_t \leqslant E[\|\hat{p}_t\| \cdot \|x_t\|] \leqslant K(y_0)\, E[\|\hat{p}_t\|]. \tag{4.6}$$

However, T is measure preserving[3] and $\hat{p}_t(s) = \hat{p}(T^t s)$ with $\hat{p} \in L^1(\phi_0)$; so $E\|\hat{p}_t\| = E\|\hat{p}\| < \infty$. Thus if we define $H = K(y_0)\, E\|\hat{p}\|$ then the claim follows from (4.5) and (4.6). ▌

LEMMA 4.5. *Suppose that* (4.1) *holds and the golden rule input,* $\hat{x}$, *is expansible. Then there is a* $0 < L < \infty$ *such that for all* $N = 0, 1, 2, \ldots,$

$$\text{“}Ep_N y_N / Ep_N \hat{x}_N \geqslant \tfrac{1}{2}\text{” implies} \qquad \text{“}Ep_N y_N \leqslant L.\text{”} \tag{4.7}$$

Proof. Let $M \geqslant 1$ be the constant in Lemma 3.5 corresponding to $\bar{b} = \frac{1}{4}$. Fix an $N = 0, 1, \ldots$. Define $b_N = Ep_N y_N / 2Ep_N \hat{x}_N$ and suppose that $Ep_N y_N / Ep_N \hat{x}_N \geqslant \frac{1}{2}$; then $b_N \geqslant \frac{1}{4}$. From Lemma 3.5 there is a feasible process $(\mathbf{x}'', \mathbf{y}'', \mathbf{c}'')$ such that $y''_0 = b_N \hat{x}_0$ and $x''_M = \hat{x}_M$. We seek to "start" the process $(\mathbf{x}'', \mathbf{y}'', \mathbf{c}'')$ at date N; to this effect we define the sequence $\{x'_t, y'_t, c'_t\}_{t=N}^{\infty}$ by $x'_t(s) = x''_{t-N}(T^N s)$, $y'_t(s) = y''_{t-N}(T^N s)$, and $c'_t(s) = c''_{t-N}(T^N s)$, for all $t \geqslant N$ and $s \in S$ (where T is the shift operator and T^N is its Nth iterate, defined in Section III.a.). Observe that $y'_N = b_N \hat{x}_N$, $x'_{N+M} = \hat{x}_{N+M}$ and for all $t \geqslant N$, $(x'_t, y'_{t+1}) = (T^N x''_{t-N}, T^N y''_{t+1-N}) \in T^t G$.

From the bound placed on the technology, assumption (T.7), there is a compact set in R^n_{++} that contains $c_t(s)$ for all $t = 0, 1, \ldots,$ and σ-almost every $s \in S$; since the utility function is continuous (U.1), we conclude that there is a $0 < J < \infty$ such that for all $t \geqslant 0$, $u(c_t(s)) \leqslant J$ a.s. Under assumptions (U.1) and (U.2) we may suppose without loss of generality that $u(0) = 0$ and $u(c) \geqslant 0$ for all $c \in R^n_+$. Hence

$$\sum_{t=N}^{N+M} \{Eu(c_t) - Eu(c'_t)\} \leqslant \sum_{t=N}^{N+M} Eu(c_t) \leqslant (M+1)\, J. \tag{4.8}$$

Next we use the competitive conditions (3.7) and (3.8) to obtain

$$\begin{aligned}
\sum_{t=N}^{N+M} \{Eu(c_t) - Eu(c'_t)\} &\geqslant \sum_{t=N}^{N+M} E\{p_t(c_t - c'_t)\} \qquad \text{(from (3.7))} \\
&= \sum_{t=N}^{N+M} E\{p_t(y_t - y'_t) - p_t(x_t - x'_t)\} \\
&\sum_{t=N}^{N+M-1} E\{p_{t+1}(y_{t+1} - y'_{t+1}) - p_t(x_t - x'_t)\} + Ep_N(y_N - y'_N) \\
&\quad - Ep_{N+M}(x_{N+M} - x'_{N+M}) \\
&\geqslant Ep_N(y_N - y'_N) - Ep_{N+M}(x_{N+M} - x'_{N+M}) \qquad \text{(from (3.8))}. \qquad (4.9)
\end{aligned}$$

Substituting $y'_N = b_N \hat{x}_N$ and $x'_{N+M} = \hat{x}_{N+M}$ in (4.9) and using the resulting expression in (4.8) yields

$$(M+1)\,J \geqslant Ep_N\, y_N - b_N\, Ep_N \hat{x}_N - Ep_{N+M}(x_{N+M} - \hat{x}_{N+M}) \quad (4.10)$$

and therefore since $b_N = Ep_N\, y_N / 2Ep_N \hat{x}_N$,

$$(M+1)\,J \geqslant \tfrac{1}{2} Ep_N\, y_N - Ep_{N+M}(x_{N+M} - \hat{x}_{N+M}). \quad (4.11)$$

But from Lemma 4.4, $Ep_{N+M}(x_{N+M} - \hat{x}_{N+M}) \leqslant H$ for some $0 < H < \infty$; using this fact in (4.11) and rearranging terms we obtain

$$Ep_N\, y_N \leqslant 2[(M+1)\,J + H]. \quad (4.12)$$

Defining $L = 2[(M+1)\,J + H]$ then concludes the proof of the lemma. ∎

We now prove Theorem 4.2.

Proof of Theorem 4.2. From [12, Theorem 1, p. 521] it suffices to show that $\sup_t Ep_t\, y_t < \infty$. Suppose, per absurdem, that this is not the case; then there is a subsequence, $\{t_k\}$, such that

$$\lim_{k \to \infty} Ep_{t_k}\, y_{t_k} = \infty. \quad (4.13)$$

We now prove the following:

Claim.

$$\limsup_{k \to \infty} Ep_{t_k} y_{t_k} / Ep_{t_k}\, \hat{x}_{t_k} \geqslant 1. \quad (4.14)$$

Proof of Claim. Fix a $t = 0, 1, \ldots$. Then

$$\frac{Ep_t\, y_t}{Ep_t \hat{x}_t} \geqslant \frac{Ep_t x_t}{Ep_t\, \hat{x}_t} = 1 + \frac{Ep_t(x_t - \hat{x}_t)}{Ep_t \hat{x}_t}. \quad (4.15)$$

From Lemma 3.3, $Ev_t \geqslant Ev_0$, where $v_t = (p_t - \hat{p}_t)(x_t - \hat{x}_t)$; a simple rearrangement of this inequality results in

$$Ep_t(x_t - \hat{x}_t) \geqslant E\hat{p}_t(x_t - \hat{x}_t) + Ev_0 \geqslant -E\hat{p}_t \hat{x}_t - |Ev_0|. \quad (4.16)$$

Since $\hat{p}_t = T^t \hat{p}$, $\hat{x}_t = T^t \hat{x}$ and T is measure preserving,[3] $E\hat{p}_t\, \hat{x}_t = E\hat{p}\hat{x}$. Substituting this in (4.16) and using the result in (4.15), we may conclude that for each $t = 0, 1, \ldots$,

$$\frac{Ep_t y_t}{Ep_t \hat{x}_t} \geqslant 1 - \frac{E\hat{p}\hat{x} + |Ev_0|}{Ep_t \hat{x}_t}. \quad (4.17)$$

Hence for each $k=0, 1, \ldots,$

$$\frac{Ep_{t_k} y_{t_k}}{Ep_{t_k} \hat{x}_{t_k}} \geqslant 1 - \frac{E\hat{p}\hat{x} + |Ev_0|}{Ep_{t_k} \hat{x}_{t_k}}. \tag{4.17)'$$

Taking the lim sup of both sides of (4.17)',

$$\limsup_{k \to \infty} \frac{Ep_{t_k} y_{t_k}}{Ep_{t_k} \hat{x}_{t_k}} \geqslant 1 - \frac{E\hat{p}\hat{x} + |Ev_0|}{\limsup_{k \to \infty} Ep_{t_k} \hat{x}_{t_k}}. \tag{4.18}$$

If $\limsup_{k \to \infty} Ep_{t_k} \hat{x}_{t_k} = \infty$, then (4.14) follows from (4.18). If, however, $\limsup_{k \to \infty} Ep_{t_k} \hat{x}_{t_k} < \infty$, then $Ep_{t_k} \hat{x}_{t_k}$ is uniformly bounded in k so (4.14) follows from (4.13). ∎

To complete the proof of Theorem 4.2, note that the claim implies that for infinitely many k's, $Ep_{t_k} y_{t_k}/Ep_{t_k} \hat{x}_{t_k} \geqslant \frac{1}{2}$; but then from Lemma 4.5, for all such k's, $Ep_{t_k} y_{t_k} \leqslant L$ for some $0 < L < \infty$. Hence

$$\liminf_{k \to \infty} Ep_{t_k} y_{t_k} \leqslant L < \infty, \tag{4.19}$$

which is a contradiction to (4.13).

References

1. L. Breiman, "Probability," Addison-Wesley, Reading, MA, 1968.
2. W. Brock, On existence of weakly maximal programmes in a multisector economy, *Rev. Econ. Stud.* **37** (1970), 275–280.
3. W. Brock and M. Majumdar, "On Characterizing Optimality of Competitive Programs in Terms of Decentralizable Conditions," Cornell University Working Paper No. 333, 1985.
4. R. Dana, Evaluation of development programs in a stationary stochastic economy with bounded primary resources, *in* "Mathematical Methods in Economics" (J. Los and M. Los, Eds.), North-Holland, Amsterdam, 1974.
5. S. Dasgupta and T. Mitra, Intertemporal decentralization in a multi-sector model when future utilities are discounted: A characterization, mimeo, 1985.
6. H. Folmer and M. Majumdar, On the asymptotic behavior of stochastic economic processes, *J. Math. Econ.* **5** (1978), 275–287.
7. D. Gale, On optimal development of a multisector economy, *Rev. Econ. Stud.* **34** (1967), 1–18.
8. L. Hurwicz, Optimality and informational efficiency in resource allocation processes, *in* "Studies in Resource Allocation Processes" (K. Arrow and L. Hurwicz, Eds.), Cambridge Univ. Press, London, 1977.

9. L. Hurwicz and M. Majumdar, "Optimal Intertemporal Allocation Mechanisms and Decentralization of Decisions," Cornell University Working Paper No. 369, 1985.
10. R. Radner, Optimal stationary consumption with stochastic production and resources, *J. Econom. Theory* **6** (1973), 68–90.
11. I. Zilcha, On competitive prices in a multi-spector economy with stochastic production and resources, *Rev. Econom. Stud.* **43** (1976), 431–438.
12. I. Zilcha, Transversality condition in a multi-sector economy under uncertainty, *Econometrica* **46** (1978), 515–525.